FUNCTIONS OF COMPLEX VARIABLE I

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These notes approximately follow Markushevich "Theory of Functions of a Complex Variable", Volumes 1, 2.

Special thanks to Misha Sergeyev for pointing out a number of typos. There are probably a few of those still remaining. **Use with caution.**

Contents

Lecture 1. Some history. Introductory nonsense

January 18, 2017.
Relevant Sections in Markushevich:

I.1.3 (partially), I.2.4, I.2.5, I.2.6.

1.1. **A bit of history.** Complex numbers did *not* appear because someone really wanted to solve the equation $x^2 = -1$. No solution, no problem, nobody cares, end of story. Instead, they came into consideration (around 16th century, Italy) when someone was interested in questions that were formulated in terms of real numbers and had answers in terms of real numbers, but required going outside of real number system in order to obtain those answers. The question was simple: solve a cubic equation $x^3 + ax^2 + bx + c = 0$. Without worrying too much about details (like whether a, b, c are real, integer, complex, or something else), let's see how the solution goes.

First of all, the substitution $x = y - \frac{a}{3}$ eliminates the quadratic term:

$$\begin{aligned} x^3 + ax^2 + bx + c &= (y - \tfrac{a}{3})^3 + a(y - \tfrac{a}{3})^2 + b(y - \tfrac{a}{3}) + c = \\ &= (y^3 - 3y^2\tfrac{a}{3} + 3y(\tfrac{a}{3})^2 - (\tfrac{a}{3})^3) + \\ &+ a(y^2 - 2y\tfrac{a}{3} + (\tfrac{a}{3})^2) + b(y - \tfrac{a}{3}) + c = \\ &= y^3 + py + q, \end{aligned}$$

with appropriate p, q. Therefore, it suffices to solve the latter cubic equation (called a depressed cubic). Make another substitution $y = \alpha + \beta$:

$$\begin{aligned} (\alpha + \beta)^3 + p(\alpha + \beta) + q &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 + p(\alpha + \beta) + q = \\ &= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) + p(\alpha + \beta) + q = \\ &= \alpha^3 + \beta^3 + (3\alpha\beta + p)(\alpha + \beta) + q. \end{aligned}$$

Since for each given y we have freedom in choosing α, β, as long as $\alpha + \beta = y$, we impose an additional condition: $3\alpha\beta + p = 0$, which gives us the equation

$$\alpha^3 + \beta^3 + 0 \cdot (\alpha + \beta) + q = \alpha^3 + \beta^3 + q = 0.$$

So if we manage to solve the system

$$\begin{cases} \alpha\beta = -p/3, \\ \alpha^3 + \beta^3 = -q, \end{cases}$$

then we can immediately express the solution as $\alpha + \beta$. Raising first equation to third power, we get a system

$$\begin{cases} \alpha^3\beta^3 = -\dfrac{p^3}{27}, \\ \alpha^3 + \beta^3 = -q \end{cases}$$

on α^3, β^3. Solutions of such system are roots of the quadratic equation $u^2 + qu - \frac{p^3}{27}$, which can be easily found:

$$\alpha^3, \beta^3 = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Recall that $y = \alpha + \beta$, so

$$y = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{1}$$

This formula is called *Cardano's* formula (it wasn't Cardano who came up with this, though).

We skipped quite a few details in the above argument, but let's just see how this works on a specific example. Consider the equation $y^3 - 15y - 4 = 0$. By the above formula we get that the number

$$\begin{aligned} y_0 &= \sqrt[3]{\frac{-4}{2} - \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}} + \sqrt[3]{\frac{-4}{2} + \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}} = \\ &= \sqrt[3]{-2 - \sqrt{-121}} + \sqrt[3]{-2 + \sqrt{-121}} = \\ &= \sqrt[3]{-2 - 11\sqrt{-1}} + \sqrt[3]{-2 + 11\sqrt{-1}} \end{aligned}$$

is a solution. This looks like some complex number. One problem, though: 4 is a root of $y^3 - 15y - 4$. Is the above actually equal to 4? Maybe this equation has some complex roots, and this is one of them? Actually, it is not hard to see that all three roots of $y^3 - 15y - 4$ are real, either by expressing $y^3 - 15y - 4 = (y-4)(y^2 - 4y + 1)$ and solving the quadratic, or by taking the derivative $(y^3 - 15y - 4)'$ and looking where local extrema of $y^3 - 15y - 4$ are. So the above expression is actually a real number.

(One might argue that we don't really need this complicated procedure, since we could just guess the root 4. However, for example, $y^3 - 6y + 2$ has three not easily guessable real roots and also has negative number under the square root in Cardano's formula.)

So here is the punchline: we asked a question about real numbers, the question has three real numbers as an answer, but to get this answer we need complex numbers. That is why complex numbers (gradually and slowly) became recognized as a valid object: despite looking really suspicious to a lot of mathematicians[1] they were successfully used to deal with questions that seemingly had anything to do only with the real number system.

1.2. **Basic Definitions. Geometry of $\mathbb{C}$.**

Definition 1. Complex numbers *are expressions of the form*

$$a + bi,$$

with $a, b \in \mathbb{R}$ *and* $i^2 = -1$*, treated as binomials* $a + bx$*, except that rule*

$$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \ldots$$

is used to eliminate any powers of i *higher than the first. Set of complex numbers is denoted by* $\mathbb{C}$.

There are other, easily equivalent, definitions that appear to be more formal.

Definition 2.

$$\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Definition 3. $\mathbb{C}$ *is a set* $\{(a,b) | a, b \in \mathbb{R}\}$ *with operations*

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

and

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

[1]Not without reason. "$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$." Huh?

Definition 4. *$\mathbb{C}$ is a quotient ring* $\mathbb{R}[x]/(x^2+1)$.

The number i is called the *imaginary unit.* For $z \in \mathbb{C}$, $z = a + bi$, the real number a is called the *real part* of z, and the real number b is called the *imaginary part* of z. Notation: $a = \operatorname{Re} z$, $b = \operatorname{Im} z$.

Regardless of which definition one prefers, one can also think of complex numbers as points or vectors on a Euclidian plane: $a + bi \leftrightarrow (a, b)$.

For a complex number $z = a + bi$, its *modulus*, or *absolute value* is the length of the corresponding vector:

$$|z| = \sqrt{a^2 + b^2}.$$

If z is a nonzero complex number, so the corresponding vector has nonzero length, denote φ to be an angle between x-axis of Euclidean plane and vector corresponding to z. Then $x = |z| \cdot \cos\varphi$ and $y = |z| \cdot \sin\varphi$. Note that such φ is only defined up to $2\pi n$. For a given $z \in \mathbb{C}$, the set of all such values of φ is called the *argument of z* and denoted $\operatorname{Arg} z$, so

$$\operatorname{Arg} z = \{\ldots, \varphi - 4\pi, \varphi - 2\pi, \varphi, \varphi + 2\pi, \varphi + 4\pi, \ldots\}.$$

An angle $\varphi \in \operatorname{Arg} z$ such that $-\pi < \varphi \leq \pi$ is called the *principal value of argument of z* and denoted $\varphi = \arg z$.

When it does not lead to ambiguity, we will abuse terminology and say $\varphi = \operatorname{Arg} z$ instead of $\varphi \in \operatorname{Arg} z$.

If $|z| = r$, $\varphi \in \operatorname{Arg} z$, we have

$$z = r(\cos\varphi + i \sin\varphi).$$

The latter expression is called the *trigonometric form* of the complex number z.

Note that for $z = r(\cos\varphi + i\sin\varphi)$, $w = s(\cos\psi + i\sin\psi)$, we have

$$zw = rs(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) = rs(\cos(\varphi+\psi) + i\sin(\varphi+\psi)). \tag{2}$$

In other words, $|zw| = |z| \cdot |w|$, and $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$. However, note that in general it is not true that $\arg(zw) = \arg(z) + \arg(w)$.

The following special case

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i \sin n\varphi \tag{3}$$

of (2) is called De Moivre's formula.

Another notation for the trigonometric form is $z = re^{i\varphi}$, where (at this point of the course) $e^{i\varphi}$ is merely a notation for

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

1.2.1. *Geometric interpretation of complex multiplication.* Note that if we fix $z_0 \in \mathbb{C}$, $z_0 \neq 0$, and multiply every complex number by z_0, the following will happen. First of all, all distances get multiplied by $|z_0|$; indeed, distance between z_1, z_2 is $|z_2 - z_1|$, and after multiplication it is $|z_0 z_2 - z_0 z_1|$, which is equal to $|z_0| \cdot |z_2 - z_1|$. Second, every line passing through origin is rotated by $\operatorname{Arg} z_0$. Therefore, geometrically multiplication by z_0 is rotation by $\operatorname{Arg} z_0$ and dilatation by a factor of $|z_0|$.

1.2.2. $\mathbb{C}$ *as a vector space.* One may also view $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$. Notice that the linear operator of multiplication by $a+bi$,

$$x+iy \mapsto (a+bi)(x+iy),$$

in the basis $1, i$ has precisely the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

which explains where Definition 2 comes from. Also, observe that for $z = a+bi$, the determinant of the corresponding matrix is $a^2+b^2 = |z|^2$, which of course was inevitable since the determinant of an operator is the coefficient by which the operator distorts the volume (in this case, area).

1.3. **Algebraic properties of $\mathbb{C}$.** $\mathbb{C}$ is a field with respect to addition and multiplication, that is, the following nine axioms hold:

(A1) $z+w = w+z$ for all $z, w \in \mathbb{C}$,
(A2) $(z+w)+u = z+(w+u)$ for all $z, w, u \in \mathbb{C}$,
(A3) there exists $0 \in \mathbb{C}$ s.t. $0+z = z+0 = z$ for all $z \in \mathbb{C}$,
(A4) for each $z \in \mathbb{C}$ there exists an element $-z$ s.t. $z+(-z) = (-z)+z = 0$,
(M1) $zw = wz$ for all $z, w \in \mathbb{C}$,
(M2) $(zw)u = z(wu)$ for all $z, w, u \in \mathbb{C}$,
(M3) there exists $1 \in \mathbb{C}$ s.t. $1 \cdot z = z \cdot 1 = z$ for all $z \in \mathbb{C}$,
(M4) for each $z \neq 0$ in $\mathbb{C}$ there exists an element $1/z$ s.t. $z \cdot (1/z) = (1/z) \cdot z = 1$,
(D) $z(w+u) = zw+zu$ and $(w+u)z = wz+uz$ for all $z, w, u \in \mathbb{C}$.

Note that Definition 1 does not change if we replace i with $j = -i$. Therefore, the mapping

$$x+yi \to x-yi$$

is an automorphism of $\mathbb{C}$ (that preserves $\mathbb{R}$). This mapping is called the *complex conjugation* and denoted $\bar{z} = x-yi$. The number $\bar{z}$ is called the complex conjugate of z. In other words, for any $z, w \in \mathbb{C}$,

$$\overline{zw} = \overline{z} \cdot \overline{w}, \qquad \overline{z+w} = \overline{z} + \overline{w}.$$

If replacing i with $-i$ is not convincing, one can, of course, check the two latter equalities explicitly.

Note that $z\bar{z} = |z|^2$, which is a non-negative real number. Using this, one can prove M4 (other properties A1–A4, M1–M3, D are either obvious, or checked straightforwardly). If $x+yi \neq 0$, that is $x^2+y^2 \neq 0$, we have

$$\frac{1}{x+yi} = \frac{x-yi}{(x+yi)(x-yi)} = \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i.$$

One can easily show that $|z|$ is actually a norm (on $\mathbb{C}$ as a vector space over $\mathbb{R}$). In particular

$$|z_1+z_2| \leq |z_1| + |z_2|$$

(if you are not comfortable with the notion of norm, this inequality equally well follows from the observation that $|z|$ is just the Euclidean the length of the corresponding vector).

Another useful thing to keep in mind is that real and imaginary parts of $z \in \mathbb{C}$ can be easily expressed through z and $\bar{z}$,

$$\operatorname{Re} z = \frac{1}{2}(z+\bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z}).$$

1.3.1. *Order on $\mathbb{C}$.* There is *no* order on $\mathbb{C}$ that agrees with arithmetic operations. In fact, there is no order that agrees with just the multiplication. Indeed, suppose we have a way of assigning some complex numbers z to be positive, denoted $z > 0$, so that two things take place:

(1) For each z, either $z = 0$, or $z > 0$, or $-z > 0$ (exclusively).
(2) For each $z_1 > 0$ and $z_2 > 0$, it follows that $z_1 z_2 > 0$.

Then either $i > 0$, so by (2) we have $-1 = i^2 > 0$, or $-i > 0$, so $-1 = (-i)^2 > 0$. But for the same reason $1 = 1^2 = (-1)^2 > 0$, so both $-1 > 0$ and $1 > 0$, contradicting (1).

(If you are curios about proper definition of order, look it up in any analysis or algebra textbook.)

1.4. **Complex roots.** Consider the equation

$$z^n = w,$$

where w is a fixed complex number, n is a positive integer, and z is the unknown. Put

$$w = R(\cos\alpha + i\sin\alpha), \quad z = r(\cos\varphi + i\sin\varphi).$$

Then by De Moivre's formula (3), or just by (2), we have

$$r^n(\cos n\varphi + i\sin n\varphi) = R(\cos\alpha + i\sin\alpha),$$

so $r = \sqrt[n]{R}$ (just an arithmetic root of a positive number), and

$$n\varphi = \alpha + 2\pi k.$$

Therefore, there are n possible values for φ that result in different solutions:

$$\varphi_j = \frac{\alpha}{n} + \frac{2\pi j}{n},$$

where j runs from 0 to $n-1$. So the set

$$\left\{ \sqrt[n]{R}(\cos\varphi_0 + i\sin\varphi_0),\ \sqrt[n]{R}(\cos\varphi_1 + i\sin\varphi_1), \dots,\ \sqrt[n]{R}(\cos\varphi_{n-1} + i\sin\varphi_{n-1}) \right\}$$

is the set of all roots of degree n of w. Note that if we try to construct a continuous square root function on $\mathbb{C}$, then as we go from $1 + \varepsilon i$ to $1 - \varepsilon i$ counterclockwise along unit circle, value of square root will change to the opposite, delivering a discontinuity.

Another issue with taking roots is apparent when we, for example, look at Cardano's formula (1). The expression in that formula has form $\sqrt[3]{A} + \sqrt[3]{B}$. At the first glance, this gives $3 \cdot 3 = 9$ different numbers, while only 3 of them are actually solutions of the original equation.

There is no easy way to fix these issues. Possible ways are:

(1) use multiple-valued functions,
(2) use so called Riemann surfaces instead of $\mathbb{C}$ as a domain,
(3) only define roots on pieces of $\mathbb{C}$ that do not "go around 0".

In this course, we will be mainly restricted to the latter option, but we will probably make a stab at some particular cases of (2) in the end of the course.

1.4.1. *Roots of* 1. *(Technically this was in Lecture 2.)* Note that all degree n roots of a number $w = R(\cos\alpha + i\sin\alpha$ can be written as

$$z_j = \sqrt[n]{R}(\cos(\tfrac{\alpha}{n} + \tfrac{2\pi j}{n}) + i\sin(\tfrac{\alpha}{n} + \tfrac{2\pi j}{n}) = z_0\xi_j,$$

where $\xi_j = \cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}$. Observe that ξ_j are degree n roots of 1. Moreover, $\xi_j = \xi^j$, where $\xi = \xi_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is the nth degree root of 1 with the smallest positive argument. So, all nth degree roots of w can be written as $z_j = z_0\xi^j$, $j = 0, \ldots, n-1$.

As an illustration of computations with roots of 1, consider all roots $1 = \xi_0$, ξ_1, $\xi_2, \ldots, \xi_{n-1}$ of degree n of 1. Let ξ be as above, then $\xi_j = \xi^j$. Find

$$1 + \xi_1 + \xi_2 + \cdots + \xi_{n-1} = S.$$

Note that $\xi S = \xi + \xi^2 + \xi^3 + \cdots + \xi^{n-1} + \xi^n = S$. So

$$\xi S = S,$$

hence $\xi = 1$ or $S = 0$. The former is impossible since ξ has nonzero argument, so $S = 0$.

Lecture 2. Stereographic projection. The extended complex plane. The mapping $1/z$

January 25, 2017
Relevant Sections in Markushevich:
I.3.9 (partially), I.3.12 (partially), I.5.20-24.

2.1. **Equation of a generalized circle.**

2.1.1. *Equation of a straight line.* Note that a vertical straight line is defined by the equation $z + \bar{z} = D$, where $D \in \mathbb{R}$. Given an arbitrary straight line, multiplying variable z by some number z_0, we get a vertical straight line in w-plane, where $w = z_0 z$. Therefore, equation of an arbitrary straight line is $zz_0 + \bar{z}\bar{z}_0 = D$. After renaming variables we get that the equation of a straight line has the form

$$E\overline{z} + \overline{E}z - D = 0., \tag{4}$$

where $E \in \mathbb{C}$, $D \in \mathbb{R}$.

2.1.2. *Equation of a circle.* Note that a circle on $\mathbb{C}$ is described by the equation

$$|z - z_0| = R.$$

Rewrite it as

$$|z - z_0|^2 = R^2,$$

or

$$(z - z_0)\overline{(z - z_0)} = R^2,$$
$$z\overline{z} - z_0\overline{z} - \overline{z_0}z + z_0\overline{z_0} - R^2 = 0.$$

Renaming variables, we get

$$z\overline{z} - E\overline{z} - \overline{E}z + D = 0,$$

where $E \in \mathbb{C}$ and $D \in \mathbb{R}$. To accommodate straight lines (4), change this equation to

$$Az\overline{z} - E\overline{z} - \overline{E}z + D = 0, \tag{5}$$

where $E \in \mathbb{C}$ and $A, D \in \mathbb{R}$. The equation (5) describes, depending on parameters, an arbitrary straight line or an arbitrary circle. Together, straight lines and circles are called *generalized circles.* To explain this choice of terminology, think of a really large circles: close up a piece of a large circle looks almost like a straight line. So, one can think of a straight line as circle that "passes through infinity". Later on we will give precise meaning to these words.

2.2. **Stereographic Projection.** Let complex plane $\mathbb{C}$ be the xy-plane in $\mathbb{R}^3$. Consider sphere Σ of radius 1 centered at origin. Denote its north pole by N and its south pole by S. Then the central projection $\sigma : \Sigma \to \mathbb{C}$ centered at N provides a bijection between Σ with N deleted and $\mathbb{C}$. (See Fig. 1.) This mapping is called the *stereographic projection.* The sphere Σ is usually called the *Riemann sphere.*

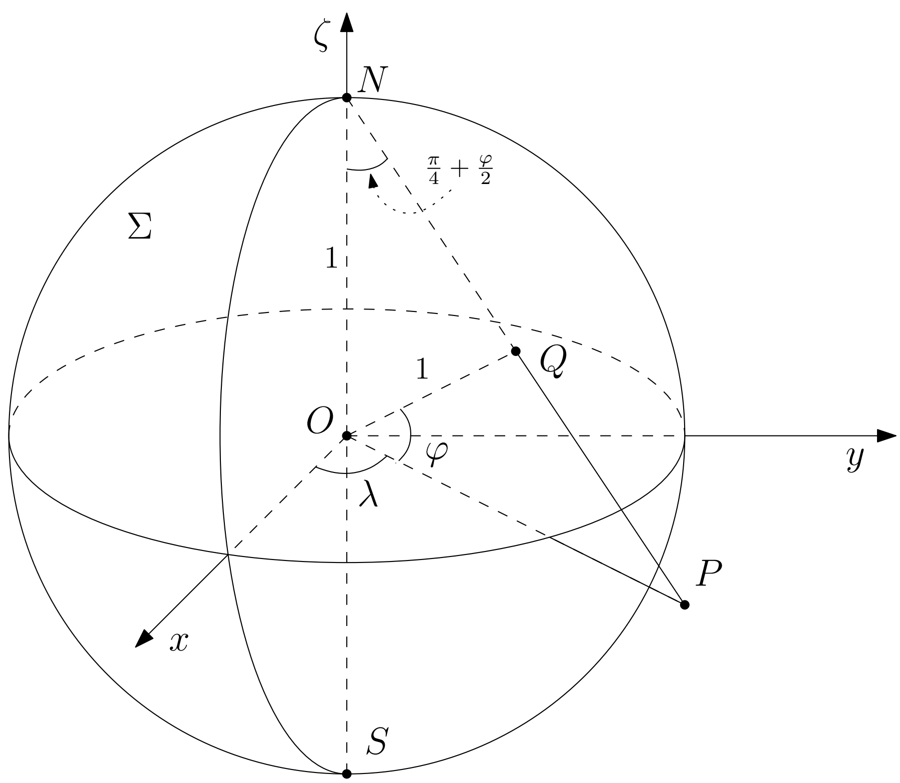

FIGURE 1. Stereographic projection.

Let $Q \in \Sigma$, and let (φ, λ) be spherical (geographic) coordinates on Σ. That is, φ is latitude, measured from equator, and λ is longitude, measured from principal meridian (the one passing through x-axis). Let P be the point on the complex plane that $Q = (\varphi, \lambda)$ is projected to. Let z be the complex number that corresponds to the point P. Then one can show that

$$z = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)(\cos\lambda + i\sin\lambda).$$

(For example, by inspecting triangle ONP and the isosceles triangle ONQ.) Also, one can see that (you might remember this from three dimensional calculus)

$$\xi = \cos\varphi\cos\lambda, \quad \eta = \cos\varphi\sin\lambda, \quad \zeta = \sin\lambda.$$

Either from inspecting triangle ONP, or from trigonometry one can see that

$$\begin{aligned} x &= \tfrac{\xi}{1-\zeta}, \\ y &= \tfrac{\eta}{1-\zeta}. \end{aligned}$$

Here, (ξ, η, ζ) are Cartesian coordinates of Q and $z = x + iy$.

It follows (for example, by squaring the equalities above and solving for ζ, taking into account $\xi^2 + \eta^2 + \zeta^2 = 1$) that

$$\begin{aligned} \xi &= \tfrac{2x}{x^2+y^2+1}, \\ \eta &= \tfrac{2y}{x^2+y^2+1}, \\ \zeta &= \tfrac{x^2+y^2-1}{x^2+y^2+1}. \end{aligned}$$

Theorem 1. *Stereographic projection preserves generalized circles. That is, image of a circle on the Riemann sphere is a circle or a straight line on the complex plane.*

Proof. Let's inspect image of a circle under stereographic projection. Note that circle on Σ is a section of Σ with a plane, that is a collection of points (ξ, η, ζ) of Σ that satisfy

$$A\xi + B\eta + C\zeta + D = 0.$$

Plugging in expressions for ξ, η, ζ through x, y, we get

$$A\frac{2x}{x^2+y^2+1} + B\frac{2y}{x^2+y^2+1} + C\frac{x^2+y^2-1}{x^2+y^2+1} + D = 0,$$

or

$$2Ax + 2Bx + (D + C)(x^2 + y^2) + (D - C) = 0$$

If $D = -C$, the latter is an equation of a straight line. Note that $D = -C$ when plane defining circle on B has equation of the form $A\xi + B\eta + C(\zeta - 1) = 0$, that is it passes through $(0, 0, 1) = N$.

If $D \neq -C$, then this is an equation of a circle. □

2.2.1. *Reminder: angle between curves.*

Definition 5. *A* curve *in $\mathbb{R}^n$ is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$.*

The image of γ is often called *curve*, too. When there is need to distinguish between different curves with the same image, different functions γ are referred to as *parameterizations.*

A curve (for simplicity, in $\mathbb{R}^3$) passing through a point P is said to *have a tangent* at P, or to be *regular* at P, if there is a parametrization $(\xi(t), \eta(t), \zeta(t))$ such that

$$(\xi'(t))^2 + (\eta'(t))^2 + (\zeta'(t))^2 \neq 0$$

at the value of parameter t corresponding to P. In this event, vector $(\xi'(t), \eta'(t), \zeta'(t))$ (again, with t that corresponds to P) is called a *tangent vector* to γ at P.

Note that differentiability of $(\xi(t), \eta(t), \zeta(t))$ at this value of t is not a sufficient condition for having a tangent.

If γ_1, γ_2 are two curves that pass through P and both have tangents at P, then by definition, angle between γ_1 and γ_2 is the angle between their tangents at P.

2.2.2. *Stereographic projection preserves angles.* Let $Q = (\xi, \eta, \zeta)$ and let γ be a curve on Σ that passes through Q and has a tangent at Q. Let $(\xi(t), \eta(t), \zeta(t))$ be an appropriate parametrization of γ. Then, as we mentioned earlier,

$$\begin{aligned} x &= \tfrac{\xi(t)}{1-\zeta(t)}, \\ y &= \tfrac{\eta(t)}{1-\zeta(t)}, \end{aligned}$$

so

$$\begin{aligned} x' &= \tfrac{\xi'(1-\zeta)+\zeta'\xi}{(1-\zeta)^2}, \\ y' &= \tfrac{\eta'(1-\zeta)+\zeta'\eta}{(1-\zeta)^2}, \end{aligned}$$

Using that $\xi^2+\eta^2 = 1-\zeta^2$ (since (ξ, η, ζ) is a point on the sphere Σ), and therefore $\xi\xi' + \eta\eta' + \zeta\zeta' = 0$, by a direct calculation we get that

$$x'^2 + y'^2 = \frac{\xi'^2 + \eta'^2 + \zeta'^2}{(1-\zeta)^2}.$$

Theorem 2. *Under stereographic projection, the angle between any two curves on the Riemann sphere Σ equals the angle between the images of the curves in the complex plane, and conversely.*

Proof. Let α be the angle between two curves $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ at point $P = (x, y)$ on a complex plane (it is of course assumed that the curves are regular (have a tangent) at P). Let β be the angle between their preimages $(\xi_1(t), \eta_1(t), \zeta_1(t))$ and $(\xi_2(t), \eta_2(t), \zeta_2(t))$ at the preimage $Q = (\xi, \eta, \zeta)$ of P under stereographic projection.

Then using expressions for x_i', y_i' and $x_i'^2 + y_i'^2$ that we got above, we obtain

$$\begin{aligned}
\cos\alpha &= \tfrac{x_1'x_2'+y_1'y_2'}{\sqrt{x_1'^2+y_1'^2}\sqrt{x_2'^2+y_2'^2}} = \\
&= \tfrac{(\xi_1'(1-\zeta)+\zeta_1'\xi)(\xi_2'(1-\zeta)+\zeta_2'\xi)+(\eta_1'(1-\zeta)+\zeta_1'\eta)(\eta_2'(1-\zeta)+\zeta_2'\eta)}{(1-\zeta)^2\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
&= \tfrac{(\xi_1'\xi_2'+\eta_1'\eta_2')(1-\zeta)^2+(\xi\xi_1'+\eta\eta_1')\zeta_2'(1-\zeta)+(\xi\xi_2'+\eta\eta_2')\zeta_1'(1-\zeta)+\zeta_1'\zeta_2'(\xi^2+\eta^2)}{(1-\zeta)^2\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
& \big(\text{using } \xi^2+\eta^2 = 1-\zeta^2 \text{ and } \xi\xi_i' + \eta\eta_i' = -\zeta\zeta_i'\big) \\
&= \tfrac{(\xi_1'\xi_2'+\eta_1'\eta_2')(1-\zeta)^2+(-\zeta\zeta_1')\zeta_2'(1-\zeta)+(-\zeta\zeta_2')\zeta_1'(1-\zeta)+\zeta_1'\zeta_2'(1-\zeta^2)}{(1-\zeta)^2\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
&= \tfrac{(\xi_1'\xi_2'+\eta_1'\eta_2')(1-\zeta)^2+\zeta_1'\zeta_2'(-\zeta(1-\zeta)-\zeta(1-\zeta)+1-\zeta^2)}{(1-\zeta)^2\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
&= \tfrac{(\xi_1'\xi_2'+\eta_1'\eta_2')(1-\zeta)^2+\zeta_1'\zeta_2'(1-\zeta)^2)}{(1-\zeta)^2\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
&= \tfrac{\xi_1'\xi_2'+\eta_1'\eta_2'+\zeta_1'\zeta_2'}{\sqrt{\xi_1'^2+\eta_1'^2+\zeta_1'^2}\sqrt{\xi_2'^2+\eta_2'^2+\zeta_2'^2}} = \\
&= \cos\beta.
\end{aligned}$$

□

Remark. In most textbooks, stereographic projection shoots the other way than in these lectures, that is from the complex plane to Riemann sphere. Does not make any substantial difference, but be aware.

Remark. Sometimes, instead of the sphere $x^2 + y^2 + z^2 = 1$, the sphere $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ is considered in the stereographic projection (and also called the Riemann sphere).

2.3. **Limits of functions and sequences in $\mathbb{C}$.** Convergence of sequences in $\mathbb{C}$ and limits of functions $\mathbb{C} \to \mathbb{C}$ carry straightforwardly from $\mathbb{R}^2$.

For example, a sequence (z_n) in $\mathbb{C}$ converges to $w \in \mathbb{C}$ if it is converges to w as a sequence in $\mathbb{R}^2$. Equivalently, we say that $\lim\limits_{n\to\infty} z_n = w$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N, |z_n - w| < \varepsilon$.

Similarly, we say that a function $f : \mathbb{C} \to \mathbb{C}$ has a limit w at z_0 if it has the same limit as a function $\mathbb{R}^2 \to \mathbb{R}^2$. Equivalently, we say that $\lim\limits_{z\to z_0} f(z) = w$ if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z,$ if $0 < |z - z_0| < \delta,$ then $|f(z) - w| < \varepsilon$.

Once limits of functions are defined, continuity of functions is defined in a usual way.

2.4. **Extended complex plane $\overline{\mathbb{C}}$.** Note that as Q approaches the north pole N on the Riemann sphere, the absolute value of image of Q approaches infinity. It is convenient to add a point ∞ to the complex plane $\mathbb{C}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and say that stereographic projection sends N to $\infty \in \overline{\mathbb{C}}$. $\overline{\mathbb{C}}$ is called the *extended complex plane*.

To be more precise, we add a new point, denoted by ∞, and explain what it has to do with other points, or, more specifically, we explain how convergence works on $\overline{\mathbb{C}}$. There is a number of ways to do that. Any of below suffices.

- Convergence on $\overline{\mathbb{C}}$ is determined by convergence on Σ via the stereographic projection. For example, for $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ converges to $w \in \overline{\mathbb{C}}$ at $z_0 \in \overline{\mathbb{C}}$ if $g = \sigma^{-1} \circ f \circ \sigma : \Sigma \to \Sigma$ converges to $\sigma^{-1}(w)$ at $\sigma^{-1}(z_0)$ (recall that $\sigma : \Sigma \to \overline{\mathbb{C}}$ is the stereographic projection).

  Similarly, (z_n) in $\overline{\mathbb{C}}$ converges to w if $\left(\sigma^{-1}(z_n)\right)$ converges to $\sigma^{-1}(w)$ on Σ.
- *(Optional.)* The same in different words: one then can think of stereographic projection as a homeomorphism between Σ and $\overline{\mathbb{C}}$, which induces topology on $\overline{\mathbb{C}}$.
- *(Optional.)* One can say that the topology on $\overline{\mathbb{C}}$ is organized by saying that open neighborhoods of ∞ are the sets of the form $\{\infty\} \cup \mathbb{C} \setminus K$, where K runs through all compact (closed and bounded) subsets of $\mathbb{C}$. This is an example of the procedure generally called *one-point compactification*.

Whichever you prefer, it gives a unified way to look at finite limits, infinite limits and limits at infinity. For example, a sequence (z_n) converges to ∞ if and only if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N, |z_n| > \varepsilon$. As another example, $f(z)$ has limit w as $z \to \infty$ if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall |z| > \delta, |f(z) - w| < \varepsilon$. All other combinations (infinite limit of a function, infinite limit at infinity) are defined similarly.

Now the stereographic projection σ is a map $\Sigma \to \overline{\mathbb{C}}$. It is a bijection (and, moreover, continuous in both directions).

We also would like to define (at least some) arithmetic operations with ∞. One way to do that is to consider the set of all convergent sequences in $\mathbb{C}$ and introduce the following equivalence relation: $(z_n) \sim (w_n)$ if $\lim(z_n) = \lim(w_n)$. Then the equivalence classes are in one-to-one correspondence with complex numbers. We add one more equivalence class: sequences that go to infinity, i.e. such sequences (z_n) that $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N, |z_n| > \varepsilon$. This new equivalence class is denoted by ∞ and called infinity.

This approach allows to easily define arithmetic operations on $\overline{\mathbb{C}}$: for $z, w \in \overline{\mathbb{C}}$ we do the needed arithmetic operation on the corresponding sequences $(z_n), (w_n)$, and if the result is the same regardless of choice of $(z_n), (w_n)$, then the operation is well-defined. This allows to define $z \cdot \infty = \infty$ (if $z \neq 0$), $z/0 = \infty$ ($z \neq 0$), $z + \infty = \infty$ ($z \neq \infty$), etc. Note that, for example, $\infty + \infty$ is not defined. Indeed, $\infty + \infty$ can be represented as $\lim((n) + (n)) = \infty$ or as $\lim((n) + (-n) = 0$.

2.5. **The mapping $\frac{1}{z}$.** Now, consider the rotation ρ of the sphere Σ about x-axis by the angle π. Under this rotation, a point with spherical coordinates (φ, λ) goes to $(-\varphi, -\lambda)$. If z corresponds to (ϕ, λ) under the stereographic projection, then for the point w that corresponds to $(-\varphi, -\lambda)$ we have (see formula in Sec. 2.2)

$$w = \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)(\cos\lambda - i\sin\lambda) = \frac{1}{\tan(\frac{\pi}{4} + \frac{\varphi}{2})(\cos\lambda + i\sin\lambda)} = \frac{1}{z}.$$

Note that this equality includes that cases $z = 0, w = \infty$ and $z = \infty, w = 0$. Therefore, denoting stereographic projection by σ, we have that the map

$$\overline{\mathbb{C}} \xrightarrow{\sigma^{-1}} \Sigma \xrightarrow{\rho} \Sigma \xrightarrow{\sigma} \overline{\mathbb{C}}$$

is precisely the complex inversion map: $z \to w = 1/z$. Since rotation of a sphere clearly preserves circles and angles, we immediately get the following statement.

Theorem 3. *Map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that sends $z \to 1/z$ preserves generalized circles (i.e. circles and straight lines) and angles between curves on the extended plane $\overline{\mathbb{C}}$.*

Note that the part about generalized circles can be easily obtained by inspecting equation (5) of a generalized circle and plugging in $z = 1/w$:

$$Az\bar{z} - \bar{E}z - E\bar{z} + D = 0,$$

$$A\tfrac{1}{w}\tfrac{1}{\bar{w}} - \bar{E}\tfrac{1}{w} - E\tfrac{1}{\bar{w}} + D = 0,$$

$$Dw\bar{w} - \bar{E}\bar{w} - Ew + A = 0.$$

Inspecting the above equation we see the following:

- Circles ($A \neq 0$) that do not pass through 0 ($D \neq 0$) go to circles that do not pass through 0.
- Circles ($A \neq 0$) that pass through 0 ($D = 0$) go to straight lines that do not pass through 0.
- Conversely, straight lines ($A = 0$) that do not pass through 0 ($D \neq 0$) go to circles that pass through 0.
- Straight lines ($A = 0$) that pass through 0 ($D = 0$) go to straight lines that pass through 0.

Lecture 3. Möbius transformations. Complex derivative. Polynomials

February 1, 2017
Relevant Sections in Markushevich:
I.7.28–30, I.8.31–34, I.9.35.

Maps that preserve angles between curves are called *conformal*. So far we've shown that the stereographic projection and $\frac{1}{z}$ are conformal maps.

3.1. **Möbius transformations.** Besides $1/z$, we consider two more conformal maps of extended complex plane:

(1) $z \to 1/z$,
(2) $z \to z + c$, $c \in \mathbb{C}$,
(3) $z \to az$, $0 \neq a \in \mathbb{C}$.

The second map is a translation, so it clearly preserves generalized circles and is conformal. The map $z \to az$ is a composition of dilatation with coefficient $|a|$ and rotation about origin by the angle $\operatorname{Arg} a$, and so it also preserves generalized circles and is conformal. Finally, note that all three maps are bijections of $\overline{\mathbb{C}}$.

Definition 6. *Maps of the form* $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, *are called* Möbius transformations *of the extended complex plane.*

Theorem 4. *Let* $a, b, c, d \in \mathbb{C}$, *and* $ad - bc \neq 0$. *Then the Möbius transformation* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *defined by*

$$f(z) = \frac{az + b}{cz + d}$$

are bijections that preserve generalized circles (i.e. circles and straight lines) and angles between curves on the extended plane $\overline{\mathbb{C}}$.

Proof. To prove this theorem, it is enough to note that

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - ad/c}{cz + d},$$

so f is a composition of maps $L_1(z) = cz + d$, $\Lambda(z) = 1/z$, $L_2(z) = \frac{a}{c} + (b - ad/c)z$:

$$f(z) = L_2(\Lambda(L_1(z))).$$

Maps L_i and Λ are of the types listed above (more exactly, L_i are easy compositions of types 2 and 3), and so f preserves circles on $\overline{\mathbb{C}}$, preserves angles between curves and is a bijection. □

Theorem 5. *Let* f, g *be Möbius transformations. Then* $f \circ g$ *and* f^{-1} *are Möbius transformations.*

Proof. (In other words, Möbius transformations form a group.) Since f and g are compositions of maps of the types (1)–(3) above, to prove this theorem, it is enough to consider $f(z) = 1/z$, $f(z) = z + c$, $f(z) = az$, which is an easy check. □

Remark. For each Möbius transformation f, we can consider a matrix whose entries are coefficients of f:

$$f(z) = \frac{az + b}{cz + d} \quad \leftrightarrow \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(There is one detail: the matrix A above is only defined up to multiplication by a constant.) One can check that taking a composition of Möbius transformations corresponds to the usual multiplication of respective matrices. In other words, if f has matrix A, and g has matrix B, one can check that the composition $f \circ g$ has matrix AB. In particular, under this observation another proof of Theorem 5 is clear, since $\det AB = \det A \det B$.

A “sciency” way to say this is that the group of Möbius transformations is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, the projective special linear group over $\mathbb{C}$, i.e., the group of

2×2 non-degenerate[2] matrices over $\mathbb{C}$, considered up to a complex multiplicative constant.

Theorem 6. *Let $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ be distinct, and let $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ be distinct. Then there exists a Möbius transformation f such that $f(z_1) = w_1,\ f(z_2) = w_2,\ f(z_3) = w_3$. Moreover, such f is unique.*

Proof. Since inverse to a Möbius transformation is a Möbius transformation, it is enough to find g such that $g(z_1) = 0, g(z_2) = 1, g(z_3) = \infty$. Note that the map

$$g(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

does exactly that. So let g be as above and h similarly send w_1, w_2, w_3 to $0, 1, \infty$, respectively. Then, by Theorem 5, $f = h^{-1} \circ g$ is the required Möbius transformation.

To show uniqueness, observe that if there two distinct Möbius transformations f_1, f_2 that send z_i to w_i, then $h \circ f_1$, $h \circ f_2$ are two distinct Möbius transformations that send z_1, z_2, z_3 to $0, 1, \infty$, respectively. Inspecting the above displayed formula, we see that such Möbius transformation is unique.

□

Remark. Based on the proof above, one can also write out explicit formula for f such that $f(z_i) = w_i$, $i = 1, 2, 3$, but it is rather long.

3.2. Complex differentiable functions. Cauchy–Riemann Equations.

Definition 7. *Let f be a complex function defined in a neighborhood of $z_0 \in \mathbb{C}$. Then f is called* complex differentiable *at z_0 if there exists limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

Definition 8. *Let f be a complex function defined in a neighborhood of $z_0 \in \mathbb{C}$. Then f is called* complex differentiable *at z_0 if f can be expressed as*

$$\Delta f = L \Delta z + \varepsilon \Delta z,$$

where $\Delta f = f(z) - f(z_0)$, $\Delta z = z - z_0$, $L \in \mathbb{C}$, and ε is a function s.t. $\varepsilon \to 0$ as $\Delta z \to 0$.

In this event, L is called the value of derivative of f at z_0 and denoted $L = f'(z_0)$, $L = \frac{df}{dz}(z_0)$.

As we know from calculus/real analysis/mathematical analysis, these two definition are equivalent (one can pass from Def. 8 to Def. 7 by simply dividing the latter displayed formula by Δz, and other way around by putting $\varepsilon = \frac{f(z)-f(z_0)}{z-z_0} - L$).

Functions that are complex differentiable on the whole complex plane $\mathbb{C}$ are called *entire.*

[*The following terms were not introduced in this lecture, we will introduce them later.* Functions, complex differentiable on an open set D, are also called *holomorphic, or analytic* on D. We say that the function f is analytic at a point z_0 if f is analytic on a neighborhood of z_0.]

Complex differentiation satisfies all normal properties of derivative:

[2]Generally, there is a difference between PSL and PGL, but not over $\mathbb{C}$.

(1) differentiation is a linear operator, that is if f, g are complex differentiable at $z_0 \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ is complex differentiable at z_0 and $(\alpha f + \beta g)' = \alpha f' + \beta g'$ at z_0.
(2) product rule takes place, that is if f, g are complex differentiable at $z_0 \in \mathbb{C}$, then so is fg and $(fg)' = f'g + fg'$ at z_0.
(3) chain rule takes place, that is if g is differentiable at $z_0 \in \mathbb{C}$, and f is differentiable at $f(z_0)$, then $f \circ g$ is differentiable at z_0 and $(f \circ g)' = f' \circ g \cdot g'$ at z_0.
(4) derivative of inverse function rule takes place, that is if one-to-one f is differentiable at z_0 and $f'(z_0) \neq 0$, then the inverse function g is differentiable at $w_0 = f(z_0)$ and $g'(w_0) = 1/f'(z_0)$.

Any function $\mathbb{C} \to \mathbb{C}$ can be viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$ by considering $f(x + iy) = u(x, y) + iv(x, y)$. The following theorem describes the difference between complex differentiability and real differentiability as a function $\mathbb{R}^2 \to \mathbb{R}^2$.

Theorem 7. (Cauchy–Riemann equations) *Let $f(z) = u(x, y) + iv(x, y)$ be a complex function of $z = x + iy$ defined in a neighborhood of z_0. Then f is complex differentiable at z_0 if and only if $u(x, y)$, $v(x, y)$ are differentiable functions of (x, y) and*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at z_0.

Proof. Prove "complex differentiable $\Rightarrow$ Cauchy-Riemann equations". Let

$$\Delta f = L\Delta z + \varepsilon \Delta z,$$

where $L = a + bi$, $\Delta z = \Delta x + i\Delta y$. Then

$$\Delta f = (a + bi)(\Delta x + i\Delta y) + \varepsilon \Delta z,$$

so

$$\Delta u + i\Delta v = (a\Delta x - b\Delta y) + i(a\Delta y + b\Delta x) + \varepsilon \Delta z,$$

Comparing Re and Im on right and left sides, we get

$$\Delta u = a\Delta x - b\Delta y + \varepsilon_1 |(\Delta x, \Delta y)|,$$

$$\Delta v = a\Delta y + b\Delta x + \varepsilon_2 |(\Delta x, \Delta y)|,$$

which exactly means that u, v are differentiable and

$$\frac{\partial u}{\partial x} = a = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -b = -\frac{\partial v}{\partial x}.$$

To prove the other direction of the theorem, one only has to perform this argument in reverse direction. □

Following the notation above, we have $f'(z_0) = a + bi$, so using expressions for a, b above we get

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

EXAMPLES

(1) $f(z) = 1$ is complex differentiable and $f'(z) = 0$.
(2) $f(z) = z$ is complex differentiable and $f'(z) = 1$.

(3) $f(z) = e^x(\cos y + i \sin y)$ is complex differentiable and $f'(z) = f(z)$. The function f is called the exponential function and denoted $f(z) = e^z = \exp(z)$. We will learn more about this function later.
(4) $f(z) = x$ is not complex differentiable anywhere (Cauchy–Riemann equations fail at every point).
(5) $f(z) = |z|^2 = z\bar{z}$ is only complex differentiable at $z = 0$ (Cauchy–Riemann equations fail at $z \neq 0$).

To elaborate on the latter two examples, note that if a complex function f is differentiable as a function $\mathbb{R}^2 \to \mathbb{R}^2$, then substituting $\Delta x = (\Delta z + \Delta\bar{z})/2$, $\Delta y = (\Delta z - \Delta\bar{z})/2i$, we can get that

$$\Delta f = A\Delta z + B\Delta\bar{z} + \varepsilon\Delta z.$$

The coefficients A, B can be viewed as formal partial derivatives by z, $\bar{z}$, respectively. One can see that f being complex differentiable is equivalent to $B = \frac{\partial f}{\partial \bar{z}} = 0$. Note that in the latter two examples we have $\frac{\partial x}{\partial \bar{z}} = \frac{\partial(z+\bar{z})}{2\partial\bar{z}} = 1/2 \neq 0$, and $\frac{\partial z\bar{z}}{\partial\bar{z}} = z$, which is 0 precisely if $z = 0$.

3.3. **Geometric interpretation of the complex derivative.**

3.3.1. *Geometric interpretation of* $\operatorname{Arg} f'$. Suppose $\lambda(t)$ is a curve $[-\varepsilon, \varepsilon] \to \mathbb{C}$ such that $\lambda'(0) \neq 0$, $\lambda(0) = z_0$. Consider function $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is a neighborhood of z_0, such that $f'(z_0) \neq 0$.

Find tangent to the curve $\Lambda(t) = f(\lambda(t))$ at $f(z_0)$:

$$\Lambda'(0) = f'(\lambda(0)) \cdot \lambda'(0) = f'(z_0) \cdot \lambda'(0).$$

Since the above is by assumption nonzero, we can consider the argument,

$$\operatorname{Arg} \Lambda'(0) = \operatorname{Arg}\left(f'(\lambda(0)) \cdot \lambda'(0)\right) = \operatorname{Arg} f'(z_0) + \operatorname{Arg} \lambda'(0).$$

In other words, *tangent to the curve* $f(\lambda)$ *at* $f(z_0)$ *is rotated by* $\operatorname{Arg} f'(z_0)$ *compared to tangent to the curve* λ *at* z_0. As a consequence, f preserves angles between curves passing through z_0.

Therefore, if $f' \neq 0$ for all points in U, then f is a conformal map $U \to \mathbb{C}$.

In particular, we can now easily establish conformity of Möbius transformations. Suppose

$$f(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. Find f':

$$f'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

so any Möbius map is conformal at every point of $\mathbb{C} \setminus \{-d/c\}$. The only question is to figure out whether f is conformal as a map of extended complex plane. That is, we need to check conformity at $z = -d/c$ and $z = \infty$.

To check the former, it suffices to check conformity of map $1/f$ at $z = -d/c$ (since transformation $1/z$ preserves angles on $\overline{\mathbb{C}}$):

$$1/f = \frac{cz + d}{az + b},$$

which is finite at $z = -d/c$ since $a(-d/c) + b \neq 0$.

To check the latter, similarly, it suffices to check conformity of map $f(1/z)$ at $z = 0$:

$$f(1/z) = \frac{a/z + b}{c/z + d} = \frac{bz + a}{dz + c},$$

which is conformal at 0 (in the event $c = 0$ we refer to the previous case).

3.3.2. *Geometric interpretation of* $|f'|$. Suppose

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and $f'(z_0) \neq 0$.

Then taking absolute value on both sides, we get

$$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|},$$

which can be interpreted the following way:

- $|f(z) - f(z_0)|$ is distance between images $f(z)$, $f(z_0)$,
- $|z - z_0|$ is distance between z, z_0,
- $\frac{|f(z)-f(z_0)|}{|z-z_0|}$ is ratio of these distances.

So $|f'(z_0)|$ is the coefficient by which f stretches distances at z_0.

Putting results of these two sections together, we get the following statement: if $f'(z_0) \neq 0$, then, in linear approximation, f at point $z = z_0$ is a composition of rotation by the angle $\operatorname{Arg} f'(z_0)$ and dilatation by $|f'(z_0)|$.

Note that if $f'(z_0) = 0$, then the linear approximation of f does not give a clear picture of behavior of f around z_0.

3.4. **Elementary functions: polynomials.** Let $P(z)$ be a polynomial of degree $n > 0$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. By fundamental theorem of algebra, P has a complex root z_1:

$$P(z_1) = 0.$$

(There is a number of proofs of this statement. We will give one in the second half of this course.) Then P can be represented as

$$P(z) = (z - z_1) Q_1(z),$$

where $\deg Q_1 = n - 1$. If, again, z_1 is a root of Q_1, repeat procedure until we get

$$P(z) = (z - z_1)^{k_1} Q(z),$$

where $\deg Q = n - k_1$ and $Q(z_1) \neq 0$. In such event, z_1 is called a root of multiplicity k_1 of P.

Terminology. Instead of "root of multiplicity k" one can also say "zero of order k". We will normally use the latter later in the course when we talk about arbitrary functions.

If $\deg Q > 0$, repeat the whole argument until we express P as

$$P(z) = a_n (z - z_1)^{k_1} (z - z_2)^{k_2} \cdots (z - z_m)^{k_m},$$

where $k_1 + k_2 + \cdots + k_m = n$, so there are at most n distinct roots.

Further, notice the following. If z_0 is a multiple root (root of multiplicity $k \geq 2$), then $P'(z_0) = 0$:

$$P'(z) = ((z - z_0)^k Q(z))' = k(z - z_0)^{k-1} Q(z) + (z - z_0)^k Q'(z),$$

so $P'(z_0) = 0 + 0 = 0$. We can say even more:

$$\begin{aligned} P'(z) &= k(z - z_0)^{k-1} Q(z) + (z - z_0)^k Q'(z) = \\ &= (z - z_0)^{k-1} (kQ(z) + (z - z_0) Q'(z)) = \\ &= (z - z_0)^{k-1} S(z), \end{aligned}$$

where $S(z_0) = kQ(z_0) + 0 \neq 0$, so z_0 is root of multiplicity exactly $k - 1$ of Q'.

Now we consider properties of P as a mapping $\mathbb{C} \to \mathbb{C}$. To start with, we find out whether a point $w \in \mathbb{C}$ has preimages under P, and if yes, how many. Fix some point $w_0 \in \mathbb{C}$. Then, to find its preimages under P, we need to find solutions of the equation

$$P(z) = w_0,$$

that is, we need to find roots of the polynomial $P(z) - w_0$. As we observed above, there are at most n distinct roots. If there fewer than n distinct roots, it means that some root z_0 has multiplicity more than 1, i.e., z_0 is a root of $(P'(z) - w_0)' = P'(z)$. Observe that $P'(z)$ does not depend on w_0 and has at most $n - 1$ roots.

Finally, recall that P is conformal at a point z whenever $P'(z) \neq 0$, so P can be not conformal only at the points which are roots of P'.

Putting the above observations to together, we get the following statement.

Lemma 1. *Degree $n > 0$ polynomial P as a map $\mathbb{C} \to \mathbb{C}$ is conformal everywhere on $\mathbb{C}$, perhaps with exception to at most $n - 1$ points $z_1, z_2, \ldots, z_m$ which are roots of P'. Moreover, for each $z \neq z_i$ (that is, for each point of conformity), preimage of $P(z)$ contains exactly n distinct elements. Preimage of each $w_i = P(z_i)$ contains fewer than n elements.*

In the next lecture, we investigate in more detail what is going on at points of non-conformity.

Lecture 4. Elementary functions: polynomials and the exponential. Basics of topology of $\mathbb{C}$

February 8, 2017
Relevant Sections in Markushevich:
I.9.35–41, partially I.3.13.

4.1. **Behavior of polynomials at points of non-conformity.**

We continue looking into polynomials as mappings $\mathbb{C} \to \mathbb{C}$.

Theorem 8. *Let z_0 be a root of multiplicity $k \geq 2$ of equation $P(z) = P(z_0)$. Then, under the mapping $w = P(z)$, every angle between curves at z_0 is enlarged k times.*

Before proving this theorem, note the following.

Remark. Let $\lambda : [-\varepsilon, \varepsilon] \to \mathbb{C}$ be a curve such that $\lambda(0) = z_0$. Then

$$(P(\lambda(t))' = P'(\lambda(t)) \cdot \lambda'(t).$$

Since we established before that $P'(z_0) = 0$, we have at $t = 0$:

$$P'(\lambda(0)) \cdot \lambda'(0) = P'(z_0) \cdot \lambda(0) = 0$$

regardless of parametrization of λ. So in order to find tangent to $P(\lambda)$ at z_0, we would have to reparametrize $P(\lambda)$ itself, which is doable but annoying. Instead, we change definition of a tangent (notice that if $\lambda'(0) \neq 0$, then the below definition is consistent with the old one).

Definition 9. *Let $\lambda : [-\varepsilon, \varepsilon] \to \mathbb{C}$ be a curve such that $\lambda(0) = z_0$. If there exists limit*

$$\theta = \lim_{t \to 0} \operatorname{Arg} \frac{\lambda(t) - \lambda(0)}{t},$$

then θ is called the angle *of tangent to λ at z_0.*

Proof of Theorem 8. First note that under the condition of the theorem,

$$P(z) - P(z_0) = (z - z_0)^k Q(z).$$

Suppose $\lambda : [-\varepsilon, \varepsilon] \to \mathbb{C}$ be a curve such that $\lambda(0) = z_0$ with tangent at z_0 at angle θ. Then taking $\Lambda(t) = P(\lambda(t))$, we get

$$\begin{gathered}
\lim_{t \to 0} \operatorname{Arg} \frac{\Lambda(t) - \Lambda(0)}{t} = \\
= \lim_{t \to 0} \operatorname{Arg} \frac{P(\lambda(t)) - P(\lambda(0))}{t} = \\
= \lim_{t \to 0} \operatorname{Arg} \frac{(\lambda(t) - z_0)^k Q(\lambda(t))}{t} = \\
= \lim_{t \to 0} \left(\operatorname{Arg} \frac{(\lambda(t) - \lambda(0))^k}{t} + \operatorname{Arg} Q(\lambda(t)) \right) = \\
= \lim_{t \to 0} \left(k \operatorname{Arg} \frac{\lambda(t) - \lambda(0)}{t} + \operatorname{Arg} Q(\lambda(t)) \right) = \\
= k\theta + \operatorname{Arg} Q(z_0).
\end{gathered}$$

Therefore, angle $\theta_2 - \theta_1$ between curves λ_1, λ_2 at z_0 gets transformed to

$$(k\theta_2 + \operatorname{Arg} Q(z_0)) - (k\theta_1 + \operatorname{Arg} Q(z_0)) = k(\theta_2 - \theta_1).$$

$\square$

Now we consider a degree $n > 1$ polynomial P as a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$. First of all,

$$\lim_{z \to \infty} P(z) = \lim_{z \to \infty} a_n z^n (1 + \tfrac{a_{n-1}}{z} + \ldots + \tfrac{a_0}{z^n}) = \lim_{z \to \infty} a_n z^n \cdot 1 = \infty,$$

so $P(\infty) = \infty$. Moreover, the angles at infinity get enlarged n times. To establish that, one only needs to consider the mapping

$$f(\zeta) = \frac{1}{P(1/\zeta)} = \frac{\zeta^n}{a_0 \zeta^n + a_1 \zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n}$$

and apply the argument above to $f = \zeta^n \cdot \frac{1}{a_0\zeta^n + \cdots + a_n}$. For consistency with the statement of Theorem 8, one may also think of ∞ as root of multiplicity n of the equation $P(z) = \infty$.

Lemma 1, Theorem 8, and the above observation together give the following statement that describes behavior of a polynomial P is a mapping $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Theorem 9. *A degree* 1 *polynomial is conformal everywhere as a map* $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.
A degree $n > 1$ *polynomial* P *as a map* $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is conformal everywhere on* $\overline{\mathbb{C}}$ *with exception to* ∞ *and at most* $n-1$ *finite points* $z_1, z_2, \ldots, z_m$ *which are roots of* P'. *For each point* z_j, *angles between curves at* z_j *are enlarged* k_j *times, where* k_j *is the multiplicity of the root* z_j *of equation* $P(z) = P(z_j)$. *Angles at* ∞ *are enlarged* n *times.*
Moreover, for each $z \neq z_i$, *preimage of* $P(z)$ *contains exactly* n *distinct elements. Preimage of each* $P(z_i)$ *contains less than* n *elements.*

As an example, consider $P(z) = z^2(z-1)^2 + 5$. We get that P is conformal except at ∞ and the roots of $P'(z) = 2z(z-1)^2 + 2z^2(z-1) = 2z(z-1)(2z-1)$, i.e., the points $z_1 = 0$, $z_2 = 1$, $z_3 = 1/2$. Since their multiplicity in P' is 1, their multiplicity in $P(z) - P(z_j)$ is 2. Therefore, angles at z_j are enlarged 2 times.

4.1.1. *Mapping* $(z-a)^n$. Consider the particular case $P(z) = (z-a)^n$. Then, by results of this section, P is conformal everywhere on $\mathbb{C}$ except at the point $z = a$. Moreover, angle θ at a gets sent to an angle $n\theta$ at 0.

4.2. **Elementary functions: the exponential.** Note that the "regular" real-valued exponential function e^x is the unique continuous function $f(x)$ that satisfies the equation $f(x_1 + x_2) = f(x_1)f(x_2)$, $f(1) = e$. Similar statement holds in case of complex valued functions.

Theorem 10. (Existence and uniqueness of the exponential) *There exists a unique single-valued complex function* f *such that the following conditions take place.*

1. $f(z) \in \mathbb{R}$ *whenever* $z \in \mathbb{R}$, *and* $f(1) = e$.
2. *For any* $z_1, z_2 \in \mathbb{C}$, f *satisfies* $f(z_1 + z_2) = f(z_1)f(z_2)$.
3. f *is complex differentiable for all* $z \in \mathbb{C}$.

Such $f(z)$ is denoted by e^z or $\exp(z)$ and called the exponential function.

Proof. (Note that there is an alternative proof that uses real-number ODE's, outlined in Homework Assignment 4.)

Suppose f is such a function.

Note that $0 \neq e = f(1) = f(z + (1-z)) = f(z)f(1-z)$, so $f(z)$ is never $= 0$. Therefore, $\operatorname{Arg} f$ and $\ln|f|$ are well-defined everywhere on $\mathbb{C}$. Taking Arg and $\ln|\cdot|$ of condition (2), we get

$$\operatorname{Arg} f(z_1 + z_2) = \operatorname{Arg} f(z_1) + \operatorname{Arg} f(z_2),$$
$$\ln|f(z_1 + z_2)| = \ln|f(z_1)| + \ln|f(z_2)|.$$

So $\operatorname{Arg} f$ and $\ln|f|$ both satisfy functional equation

$$F(z_1 + z_2) = F(z_1) + F(z_2),$$

with the following difference: $\ln|f|$ is a single-valued function, $\operatorname{Arg} f$ is a multiple valued function, defined up to $2\pi j$, $j \in \mathbb{Z}$. Further, split $z = x + iy$ and use the equation above:

$$F(x + iy) = F(x) + F(iy).$$

Therefore, the functional equation above is satisfied by the following *real functions of real argument*:

$$\operatorname{Arg}\, f(x), \operatorname{Arg}\, f(iy),$$
$$\ln|f(x)|, \ln|f(iy)|.$$

Arg is still multiple valued, though.

Solve $F(x_1+x_2) = F(x_1)+F(x_2)$ for a single valued real F. One can easily see that in this case F is linear. Indeed, assume $F(1) = a$. Then

$$nF(1/n) = F(1/n)+F(1/n)+\cdots+F(1/n) = F(1/n+1/n+\cdots+1/n) = F(1) = a,$$

so $F(1/n) = a/n$ for any positive integer n. Also,

$$F(m/n) = F(1/n+\cdots+1/n) = mF(1/n) = am/n$$

for any positive integers m, n. Further, $F(0)+F(0) = F(0+0) = F(0)$, so $F(0) = 0$, so $F(-x)+F(x) = F(0) = 0$, therefore $F(x) = ax$ for any *rational* x. Since F is continuous, it immediately follows that $F(x) = ax$ for any $x \in \mathbb{R}$.

In case of multiple valued function F defined up to $2\pi j$, one can establish that $F(x) = \alpha x + 2\pi j$, where $j \in \mathbb{Z}$. (Key point is to observe that with "small" values of x_1, x_2, the equality $F(x_1+x_x) = F(x_1)+F(x_2)$ holds as is, rather than up to $2\pi k$. EXERCISE: work out the details.)

Therefore, we have

$$\operatorname{Arg} f(x) = \alpha x + 2\pi j, \quad \operatorname{Arg} f(iy) = \beta y + 2\pi l,$$

$$\ln|f(x)| = ax, \quad \ln|f(iy)| = by,$$

so

$$\operatorname{Arg}\ f(z) = \alpha x + \beta y + 2\pi k, \quad \ln|f(x)| = ax + by.$$

Use condition (1): since $f(x)$ is real, we immediately conclude that $\alpha = 0$, and since $f(1) = e$, we immediately conclude $a = 1$. Then

$$f = |f|(\cos \operatorname{Arg} f + i \sin \operatorname{Arg} f) =$$

$$f = e^{x+by}(\cos \beta y + i \sin \beta y).$$

Now make sure that condition (3) is satisfied, that is check Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = e^{x+by}\cos \beta y$$

$$\frac{\partial v}{\partial y} = be^{x+by}\sin \beta y + \beta e^{x+by}\cos \beta y$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ at all points z, we conclude that $b = 0$, $\beta = 1$. A quick check shows that in such case the other Cauchy–Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ is also satisfied. Therefore, the function f satisfies (1)–(3) if and only if

$$f(z) = f(x+iy) = e^x(\cos y + i \sin y).$$

□

Note that the condition (3), redundant in the real case, is essential for uniqueness here: there are multitude (given by any values of b and β other than 0 and 1 respectively) of complex non-differentiable functions that are nevertheless continuous, real differentiable and satisfy conditions (1), (2) of the theorem.

4.3. **Some properties of the exponential.** Recall that $e^z = e^x(\cos y + i \sin y)$, where $z = x + iy$. In particular, e^z is never 0.

Now find derivative of e^z. To do that, recall that from the proof of Cauchy–Riemann equations (Theorem 7), we have that if $f'(z) = a + bi$, then

$$a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. So, for example,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

So for $f = e^z$ we have

$$f'(z) = e^x \cos y + ie^x \sin y = e^z.$$

In particular, for each $z \in \mathbb{C}$, $f'(z) \neq 0$. *Therefore, $z \to e^z$ is a conformal mapping $\mathbb{C} \to \mathbb{C}$.*

What about $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$? Turns out, exp cannot be extended continuously to $\overline{\mathbb{C}}$. Indeed, note that $\lim\limits_{x \to +\infty} e^x = \infty$, while $\lim\limits_{x \to -\infty} e^x = 0$, so the limit $\lim\limits_{z \to \infty} e^z = \infty$ does not exist. This also explains that exp is not a polynomial: if it was, it would be ∞ at ∞.

Stare a bit harder at how this map works. To start with, we take a look at the image of horizontal and vertical lines. Parametric equation of a horizontal line is $t + ib$, $t \in \mathbb{R}$.

$$t + ib \to e^{t+ib} = e^t(\cos b + i \sin b),$$

which is a ray originating at 0 at the angle b to real axis.

Parametric equation of a vertical line is $a + it$, $t \in \mathbb{R}$.

$$a + it \to e^{a+it} = e^a(\cos t + i \sin t),$$

which is a circle of radius e^a. So, the grid of horizontal and vertical lines is sent to the "grid" of rays originating at 0 and circles centered at 0. Note that this presents a very apparent illustration to conformity of $z \to e^z$, since two orthogonal systems of lines are sent to two orthogonal systems of curves.

Now, we look at the image of a horizontal strip $0 \leq \operatorname{Im} z \leq h$. As we know from the consideration above, the line $\operatorname{Im} z = 0$ is sent to a ray originating at 0 at the angle 0 to real axis, and the line $\operatorname{Im} z = h$ is sent to a ray originating at 0 at the angle h to real axis. Then the interior of the strip gets sent to interior of the angle formed by these two rays. Note in particular that if $h \geq 2\pi$, then the image is $\mathbb{C} \setminus \{0\}$.

Finally, find periods of the exponential. Note that for any $z \in \mathbb{C}$, we have $e^{z+2\pi i} = e^z$, so any number $2\pi k i$, $k \in \mathbb{Z}$, is a period of the exponent. Make sure that there are no other periods. Suppose w is a period of exponent. Then $e^{z+w} = e^z$ for any $z \in \mathbb{C}$. Since $e^z \neq 0$, divide both sides by e^z and get $e^w = 1$. It is easy to see that solutions of this equation are exactly the numbers $w = 2\pi k i$, $k \in \mathbb{Z}$.

4.3.1. *Similarity between* exp *and* $(z-a)^n$ *as* $n \to \infty$. This section is optional, as it was not included in the lecture.

If we are looking at the image of a horizontal strip under exp, its left end gets "squished" to 0, while its right end gets "stretched" to ∞. This bears similarity with map $z \to (z-a)^n$. Examine this similarity in more detail. Recall first that, as we know from calculus, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty}(1+x/n)^n = e^x.$$

One can prove (see Homework Assignment 4) that the same holds for complex numbers: for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty}(1+z/n)^n = e^z,$$

so the map $z \to (1+z/n)^n$ can be thought of as approximations of $z \to e^z$. Now write

$$(1+z/n)^n = n^{-n}(z-(-n))^n.$$

Consider the image under $z \to (z-(-n))^n$ of an angle formed by real axis and a ray originating at $-n$ at the angle h/n to real axis. (We ignore coefficient n^{-n} because is a positive real number, and therefore does not change arg.) This angle gets sent to the angle $n \cdot h/n = h$ at 0, just like the strip of height h under the exponential map. Moreover, the ray at the angle h/n at $-n$ intersects imaginary axis at the point $in\tan h/n$. As $n \to \infty$, $in\tan h/n \to nh/n = h$, so this angle looks more and more like a strip of height h.

4.4. **Functions related to the exponential.** Note that if we put $z = iy$, $y \in \mathbb{R}$, in the formula for e^z, we see that

$$\cos y = \frac{e^{iy}+e^{-iy}}{2}, \quad \sin y = \frac{e^{iy}-e^{-iy}}{2i}.$$

For an *arbitrary* $z \in \mathbb{C}$, *define* the trigonometric functions sine and cosine to be

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

These definitions extend real sine and cosine functions, and bear many of the same properties:

$$\cos(-z) = \cos z, \quad \sin(-z) = -\sin z,$$

as easily seen from definition. Moreover,

$$\cos^2 z + \sin^2 z = 1,$$

(again, immediately from definition). Note, however, that cos and sin are no longer bounded.

One can also check (similarly to how it's done in the case of the exponential) that sin and cos are periodic with periods $w = 2\pi k$, $k \in \mathbb{C}$.

Further, considering $z = z_1 + z_2$, we get that

$$\begin{aligned}\cos(z_1+z_2) \quad + \quad & i\sin(z_1+z_2) = \exp(i(z_1+z_2)) = \exp(iz_1)\exp(iz_2) = \\ = \quad & (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) = \\ = \quad & (\cos z_1\cos z_2 - \sin z_1\sin z_2) + i(\cos z_1\sin z_2 + \sin z_1\cos z_2),\end{aligned}$$

and

$$\begin{aligned}\cos(z_1+z_2) \quad - \quad & i\sin(z_1+z_2) = \exp(-i(z_1+z_2)) = \exp(-iz_1)\exp(-iz_2) = \\ = \quad & (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2) = \\ = \quad & (\cos z_1\cos z_2 - \sin z_1\sin z_2) - i(\cos z_1\sin z_2 + \sin z_1\cos z_2),\end{aligned}$$

adding and subtracting one equation from another, we establish formulas

$$\cos(z_1+z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2, \tag{6}$$

$$\sin(z_1 + z_2) = \cos z_1 \sin z_2 + \sin z_1 \cos z_2. \tag{7}$$

Observe that one property of cosine and sine that is not carried from the case of real numbers is boundedness. While $|\cos x|, |\sin x| \leq 1$ for $x \in \mathbb{R}$, it is easy to see that $|\cos z|$ and $|\sin z|$ are unbounded if $z \in \mathbb{C}$.

Closely related to $\exp, \sin, \cos$, are the *hyperbolic functions* $\cosh z$ and $\sinh z$, called the hyperbolic cosine and hyperbolic sine, respectively. The are defined by formulas

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

So,

$$\cosh z = \cos(iz), \quad \sinh z = -i \sin(iz).$$

In particular, for example, $\cosh^2 z - \sinh^2 z = 1$. Any further information about cosh and sinh can be derived from expression through the exponential or cosine and sine.

Finally, we mention that once $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are defined, we can define $\tan z$, $\cot z$, $\tanh z$, $\coth z$ in the usual way. Their properties can also be deduced either from their definitions through (hyperbolic) sine and cosine, or from their expressions through the exponential function.

4.5. **Notion of elementary functions.** Complex functions built from a finite number of exponentials, polynomials, and inverses to exp (called logarithm) and z^n (called nth roots) through composition and combinations using the four arithmetic operations, are collectively called *elementary functions*. We will discuss the logarithm and the nth root in detail later in the course.

Note that this class, in particular, includes all trig functions. (Exercise for later in the course, when we introduce ln properly: express arcsin through ln).

Finally, recall that functions $\mathbb{C} \to \mathbb{C}$ that are complex differentiable everywhere on $\mathbb{C}$ are called entire. So far we have two main classes of examples: polynomials and the exponential (and some of the related functions, like cos, but not, say, tan).

4.6. **Basic notions of topology of $\mathbb{C}$.** In this section we start covering a bare minimum of topological terms needed to proceed to theory of complex integral. Note that the definitions below are given in relation to $\mathbb{C}$. General definitions are different is some cases. If you have difficulties with these, you can find them in any Topology (and most Analysis) textbook (for example *Topology* by Munkress or *Elementary Topology. Problem Texbook* by Viro et all).

An *open disc* $B_r(z_0)$ of radius $r > 0$ centered at z_0 is the set

$$\{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

We will often refer to an open disc $B_r(z_0)$ as a *neighborhood* of z_0, or an r-neighborhood of z_0.

A *closed disc* of radius $r \geq 0$ centered at z_0 is the set

$$\{z \in \mathbb{C} \mid |z - z_0| \leq r\}.$$

A subset $U \subseteq \mathbb{C}$ is called *open* if for any point $z_0 \in U$, there is a neighborhood of z_0 contained in U: $\forall z_0 \in U\ \exists B_r(z_0) \subseteq U$.

Examples: open disc is an open set, $\mathbb{C}$ is open set, $\emptyset$ is an open set, a single point $\{z_0\}$ is not an open set, a straight line is not an open set, an open interval $(0,1) \subseteq \mathbb{C}$ is not an open set.

For a set $E \subseteq \mathbb{C}$, the *boundary* of E is the set ∂E that consists of points $z_0 \in \mathbb{C}$ such that any neighborhood of z_0 contains both points from E and from its complement $\mathbb{C} \setminus E$.

Examples: boundary of an open disc is the corresponding circle, boundary of a closed disc is the corresponding circle, boundary of $\mathbb{C}$ is empty, boundary of $\mathbb{Q} \subseteq \mathbb{C}$ is $\mathbb{R}$.

A subset $C \subseteq \mathbb{C}$ is called *closed* if $\partial C \subseteq C$. Alternatively, set C is closed if and only if its complement $\mathbb{C} \setminus C$ is open. Vice versa, set C is open if and only if its complement $\mathbb{C} \setminus C$ is closed. Proving these two statements is a nice exercise in applying definitions.

For a subset $E \subseteq C$, its *closure* is the set $\overline{E} = E \cup \partial E$. Exercise: closure of a set is closed set.

A fundamental property of open and closed sets is the following:

- Any union of open sets is open.
- Any finite intersection of open sets is open.
- Any finite union of closed sets is closed.
- Any intersection of closed sets is closed.

In the next lecture, we will continue with topological notions, such as connectedness, simple connectedness, interior, exterior, Jordan curve theorem.

Lecture 5. More topology. Definition of complex integral

February 15, 2017
Relevant Sections in Markushevich:
I.4.15–18, I.12.60–62.

5.1. **More topology.** We continue with introducing notions and terminology that will be later used when dealing with complex integral.

A subset $E \subseteq C$ is called *connected* if the following holds: given any decomposition of E into two nonempty disjoint sets E_1, E_2, at least one of the sets E_1, E_2 contains a point of closure of the other. In other words, if

$$E_1, E_2 \neq \emptyset, \quad E_1 \cap E_2 = \emptyset, \quad E_1 \cup E_2 = E, \tag{8}$$

then $\overline{E_1} \cap E_2 \neq \emptyset$ or $E_1 \cap \overline{E_2} \neq \emptyset$.

$\big[$If you are interested in a bit more topological detail, there is a notion of a set *open in a set* X. Let X be a fixed subset of $\mathbb{C}$. Then a subset $U \subseteq X$ is called *open in* X if $U = X \cap \mathcal{O}$, where $\mathcal{O}$ is an open subset of $\mathbb{C}$. Examples: X is always open in itself, $[0, 1/2) \subseteq \mathbb{C}$ is open in $[0, 1]$. One can show that a function $X \to \mathbb{Y}$ (where $X, Y \subseteq \mathbb{C}$) is continuous if and only if preimage of every set open in Y is is open in X. Checking that this condition is equivalent to the ε–δ definition is a nice exercise.

In these terms, E is connected if under the same conditions (8), E_1 and E_2 cannot be both open in E.$\big]$

A subset $E \subseteq C$ is called *path (pathwise, arcwise, linearly) connected* if for any two points $z_1, z_2 \in E$, there is a curve that joins them, i.e. a curve $\gamma : [a, b] \to E$ such that $\gamma(a) = z_1, \gamma(b) = z_2$.

Example: a subset $E = \{(x, \sin(1/x)) \mid x \in (0, 1)\} \cup \{(0, t) \mid -1 \leq t \leq 1\}$ is connected but not path connected.

Theorem 11. *If a subset $E \subseteq \mathbb{C}$ is path connected, it is connected.*

Proof. (Sketch.) If $E = E_1 \cup E_2$ is a "disconnecting" decomposition, pick a point $z_1 \in E$ and $z_2 \in E_2$. Let $\gamma : [a, b] \to E$ be a curve that connects z_1 to z_2. Under γ^{-1}, the decomposition $E = E_1 \cup E_2$ induces a "disconnecting" decomposition of $[a, b]$. $\square$

Theorem 12. *If an* open *subset $E \subseteq \mathbb{C}$ is connected, it is path connected.*

Proof. (Sketch.) Fix a point $z_0 \in E$. Note that both sets

$$E_1 = \{z \in E \mid z \text{ can be reached by a curve from } z_0\}$$

and

$$E_2 = \{z \in E \mid z \text{ cannot be reached by a curve from } z_0\}$$

are open. If E is not path connected, then $E_1, E_2 \neq \emptyset$, which delivers a decomposition (8) with E_1, E_2 open. $\square$

Proofs of these two theorems are, strictly speaking, outside this course. However, they are quite instructive exercises in application of notions introduced in this section, so you are encouraged to work your way through these theorems.

Every set $E \subseteq \mathbb{C}$ can be represented as a union of disjoint connected sets. Those connected sets are called the connected components of E. For example, the set $E = \{z : |z| < 1 \text{ or } |z - 3| < 1\}$ has two connected components, the disks $E_1 = \{z : |z| < 1\}$ and $E_2 = \{z : |z - 3| < 1\}$.

Definition 10. *A connected open subset of $\mathbb{C}$ is called a* domain.

Note that the above is not a common definition, but this type of sets is used in the course so often, that we give it a separate name.

Definition 11. *Let $E \subseteq \mathbb{C}$. Then the* set of interior points of E *(or just the* interior of E*) is the set*

$$\mathrm{Int}(E) = \{z \in E \mid \exists B_r(z) \subseteq E\}.$$

The set of exterior points of E *(or just the* exterior of E*) is the set*

$$\mathrm{Ext}(E) = \{z \in \mathbb{C} \mid \exists B_r(z) \cap E = \emptyset\}.$$

Examples:

(1) For $E = B_r(z_0)$, $\mathrm{Int}(E) = B_r(z_0)$, $\mathrm{Ext}(E) = \{z \in \mathbb{C} \mid |z - z_0| > r\}$.
(2) For $E = \mathbb{Q} \subseteq \mathbb{C}$, $\mathrm{Int}(E) = \emptyset$, $\mathrm{Ext}(E) = \{z \in \mathbb{C} \mid \mathrm{Im} z \neq 0\}$.

Note that for any set $E \subseteq C$, $\mathbb{C} = \mathrm{Int}(E) \cup \mathrm{Ext}(E) \cup \partial E$. (The boundary ∂E of a set E was defined in the previous lecture.)

Also note that for any $E \subseteq \mathbb{C}$, both $\mathrm{Ext}(E)$ and $\mathrm{Int}(E)$ are open (immediate consequence of definition of open set). In particular, if E is open, then $\partial E = \mathbb{C} \setminus (\mathrm{Ext}(E) \cup \mathrm{Int}(E))$ is closed. (This, of course, can be also seen directly.)

Definition 12. *A subset $E \subseteq \mathbb{C}$ is called* bounded *if $\exists B_r(z_0)$ such that $E \subseteq B_r(z_0)$. Equivalently, $\exists M \in \mathbb{C}$ s.t. $\forall z \in E \quad |z| < M$.*

Definition 13. *A bounded domain G is called* simply connected *if ∂G is a connected set. A bounded domain G is called* n-connected *(or* multiply connected*) if ∂G has n connected components.*

Beware: the definition above is one of the few here that *do not translate* "as is" to the case of arbitrary topological spaces. In fact, it does not even translate immediately to the case of unbounded domains.

Below we state two theorems which we will use later on but which are, unfortunately, too annoying to prove within this course (especially the first one).

A curve $\gamma : [a, b] \to \mathbb{C}$ is called *simple* if it does not self-intersect except maybe at the endpoints: if $\gamma(x) = \gamma(y)$ then $x = y$ or $\{x, y\} = \{a, b\}$. A curve $\gamma : [a, b] \to \mathbb{C}$ is called closed if $\gamma(a) = \gamma(b)$. A closed curve can be thought as a mapping from a circle (rather than an interval) to $\mathbb{C}$. A simple closed curve is called a *Jordan* curve. In these notes, we will mostly keep calling such curves "simple closed".

Theorem 13. (Jordan curve theorem) *Complement $\mathbb{C} \setminus \gamma$ of a closed simple curve γ has exactly two connected components, with γ as their common boundary. One of these components, called the* interior *of γ and denoted $I(\gamma)$, is bounded. The other component, called the* exterior *of γ and denoted $E(\gamma)$, is unbounded.*

Theorem 14. *A bounded domain G is simply connected if and only if whenever G contains a simple closed curve γ, domain G also contains $I(\gamma)$.*

If we were to define simple connectedness in case of unbounded domains, the above theorem would still be true for those. This theorem is what we will be using practically when we later look into behavior of integral along closed curves.

5.2. **Definition of complex integral.** Let $[a, b]$ be a closed interval on a real line. Let t_k, $k = 0, \ldots, n$ be as follows:

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b.$$

In such event, the collection $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ is called a *partition* of $[a, b]$. The real number

$$|\mathcal{P}| = \max\{t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1}\}$$

is called the *norm* (or *diameter*) of the partition $\mathcal{P}$.

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous curve, and let $\mathcal{P}$ be a partition as above. Denote $\gamma(t_k) = z_k$, $i = 0, \ldots, n$. If

$$\sup_{\mathcal{P}} \sum_{k=1}^{n} |z_k - z_{k-1}| = l < \infty,$$

then the curve γ is called *rectifiable* and l is called the *length of* γ.

A curve $\gamma : [a, b] \to \mathbb{C}$ is called *smooth* if there exists a parametrization of γ such that $\gamma'(t)$ is continuous and nowhere zero. A curve $\gamma : [a, b] \to \mathbb{C}$ is called *piecewise smooth* if there is a partition of $[a, b]$ such that γ is smooth on every interval of the partition.

Here is a fact from calculus: if $\gamma = \mu + i\nu$ is a smooth curve, then

$$l = \int_a^b \sqrt{(\mu')^2 + (\nu')^2}dt = \int_a^b |\gamma'(t)|dt.$$

Let be $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. Let collection $\{\tau_k\}$, $k = 1, \ldots, n$ be such that $t_{k-1} \le \tau_k \le t_k$ for every $k = 1, \ldots, n$. Then τ_k's are called *tags* and pair

$$\dot{\mathcal{P}} = (\{t_0, t_1, \ldots, t_n\}, \{\tau_1, \ldots, \tau_n\})$$

is called a *tagged partition* of $[a, b]$. We define $|\dot{\mathcal{P}}| = |\mathcal{P}|$.

Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable curve, let $\dot{\mathcal{P}}$ be a tagged partition of $[a, b]$. Denote $\gamma(t_k) = z_n$ and $\gamma(\tau_k) = \zeta_k$. Let f be a complex function defined at every point of γ. Then the *Riemann sum of f along γ with respect to partition $\mathcal{P}$* is the following sum:

$$S(f, \dot{\mathcal{P}}) = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}).$$

If the limit[3]

$$I = \lim_{|\dot{\mathcal{P}}| \to 0} S(f, \dot{\mathcal{P}})$$

exists, function f is called *integrable along* γ and the number I is called *integral of f along* γ (or *the value of integral of f along* γ), denoted

$$I = \int_\gamma f(z) dz.$$

Before continuing with properties of complex integral, consider Riemann sums in more detail. Denote $f(z) = u + iv$, $f(\zeta_k) = u_k + iv_k$, and $z_k - z_{k-1} = \Delta x_k + i\Delta y_k$. Then

$$\begin{aligned}
\textstyle\int_\gamma f(z)dz &= \lim_{|\dot{\mathcal{P}}| \to 0} \textstyle\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \\
&= \lim_{|\dot{\mathcal{P}}| \to 0} \textstyle\sum_{k=1}^{n} (u_k + iv_k)(\Delta x_k + i\Delta y_k) = \\
&= \lim_{|\dot{\mathcal{P}}| \to 0} \textstyle\sum_{k=1}^{n} ((u_k \Delta x_k - v_k \Delta y_k) + i(u_k \Delta y_k + v_k \Delta x_k)) = \\
&= \lim_{|\dot{\mathcal{P}}| \to 0} \textstyle\sum_{k=1}^{n} (u_k \Delta x_k - v_k \Delta y_k) + i \lim_{|\dot{\mathcal{P}}| \to 0} (u_k \Delta y_k + v_k \Delta x_k) = \\
&= \textstyle\int_\gamma u\, dx - v\, dy \quad + \quad i \int_\gamma v\, dx + u\, dy,
\end{aligned}$$

where the two latter integrals are "usual" real valued curve integrals. Therefore, we reduced complex integral to a special case of curve integrals. In particular, now we don't need to prove many of the properties of complex integral, but rather refer to the real case.

One important implication is the following. It is known from calculus that if $\gamma = (\mu, \nu) : [a, b] \to \mathbb{R}^2$ is a smooth curve in $\mathbb{R}^2$, then

$$\int_\gamma P(x, y)dx + Q(x, y)dy = \int_a^b (P(\mu(t), \nu(t))\mu'(t) + Q(\mu(t), \nu(t))\nu'(t))dt.$$

if the integral in the left hand side exists. Applying this to the integrals in the last line of the chain of equalities in the previous paragraph, and performing the same computation "bottom to top", we get an important practical formula: if $\gamma = \mu + i\nu : [a, b] \to \mathbb{C}$ is a smooth curve and f is integrable along γ, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{9}$$

The advantage of this expression is that complex integral along a curve is reduced to the "usual" Riemann integrals of real and imaginary parts of the integrand on the right hand side.

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[3]This is not a limit of a function, because $S(f, \dot{\mathcal{P}})$ is not a function of $|\dot{\mathcal{P}}|$. Proper way would be to define it through ε–δ machinery in a usual way. We don't do it here, because all proofs of basic properties remain the same in either case. By the way, there is the same issue with the "usual" real-valued Riemann integral. Another proper way of fixing this is using limits of *nets*.

5.3. **Examples and basic properties of complex integral.** Now we give some examples of complex integrals, computed by the definition and using formula (9).

- Integral of a constant. $f(z) = 1$, $\gamma : [a, b] \to \mathbb{C}$ arbitrary s.t. $\gamma(a) = z_0$, $\gamma(b) = Z$. Compute the integral by definition. Let $\dot{\mathcal{P}}$ be arbitrary. We have
$$\begin{aligned}\textstyle\int_\gamma 1dz &= \lim_{|\dot{\mathcal{P}}|\to 0} \textstyle\sum_{k=1}^n 1 \cdot (z_k - z_{k-1}) = \\ &= \lim_{|\dot{\mathcal{P}}|\to 0} z_n - z_0 = Z - z_0,\end{aligned}$$
as expected. In particular, the integral does not depend on a specific choice of γ.
- Integral of a constant. Assume additionally that γ is smooth and use formula (9):
$$\int_\gamma 1dz = \int_a^b 1 \cdot \gamma'(t)dt = \gamma(t)\big|_a^b = Z - z_0.$$
Notice that the middle equality is obtained by applying real-value Fundamental Theorem of Calculus to Re and Im separately.
- Integral of a linear function $f(z) = z$. We know from the reduction to the real case that linear functions are integrable along any rectifiable curves. Therefore, to find the value of the integral, we can make a *particular choice* of partitions and tags and take a limit. In a partition $\mathcal{P}$, choose tags in two ways: to be the left endpoints of partition intervals, and the right endpoints of partition intervals:
$$\int_\gamma zdz = \lim_{|\dot{\mathcal{P}}|\to 0} z_{k-1}(z_k - z_{k-1}), \text{ and } \int_\gamma zdz = \lim_{|\dot{\mathcal{P}}|\to 0} z_k(z_k - z_{k-1}).$$
Taking half-sum of these two equalities, we observe that most terms in the right hand side cancel, so we get that
$$\int_\gamma zdz = \frac{Z^2}{2} - \frac{z_0^2}{2},$$
as expected. And again, integral does not depend on a specific choice of path.
- Integral of a linear function. Assume additionally that γ is smooth and use formula (9):
$$\int_\gamma zdz = \int_a^b \gamma(t) \cdot \gamma'(t)dt = \tfrac{1}{2}\, \gamma^2(t)\big|_a^b = \frac{Z^2}{2} - \frac{z_0^2}{2}.$$
As we plainly see in this example, computing integrals using Riemann sums is more painful.

  Notice that the middle equality above is not completely obvious. To justify it, we differentiate Re and Im of γ^2 and apply real-value Fundamental Theorem of Calculus to them separately.
- Integral of $\frac{1}{z}$. Consider $f(z) = \frac{1}{z-a}$, $a \in \mathbb{C}$, and γ a circle $z = a + re^{it}$ traversed once, that is $0 \le t \le 2\pi$. Then
$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{(a + re^{it})'dt}{a + re^{it} - a} = \int_0^{2\pi} \frac{ire^{it}dt}{re^{it}} = \int_0^{2\pi} idt = 2\pi i.$$

This gives an example of integral that depends on a choice of path of integration. Note, however, that in this particular example, the integral does not depend on radius of the circle of integration.

In HW5, we do the same computation through Riemann sums.

Now that we have seen some examples, we continue with basic properties of complex integral.

(1) Additivity over curves:
$$\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz,$$
where by $-\gamma$ we mean the same curve γ, traversed in opposite direction.

(2) Additivity over curves:
$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz,$$
where γ is concatenation of $\gamma_1, \gamma_2, \ldots, \gamma_n$, denoted $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

Note that the equality can be read both ways: assuming γ is subdivided into $\gamma_1 + \cdots + \gamma_n$, if either left side or right hand side of the above equality exists, then the other side exists, too, and they are equal.

(3) Linearity:
$$\int_\gamma \sum_{k=1}^n c_k f_k(z)dz = \sum_{k=1}^n \int_\gamma c_k f_k(z)dz,$$
where f_k are functions defined on and integrable along γ, and $c_k \in \mathbb{C}$.

(4) Boundedness: If f is integrable along γ, $|f(z)| \le M$ for all $z \in \gamma$, and l is the length of γ, then
$$\left|\int_\gamma f(z)dz\right| \le Ml.$$

Properties (1), (3), (4) can be seen immediately from the definition of integral (since corresponding statements hold when the integrals are replaced by their Riemann sums). Property (2) also can be derived from the definition, but requires some handwork. Instead, we just say that it holds since it holds for the respective curve integrals, as was proven in calculus course.

Note that the properties above are an immediate translation of the corresponding properties of the real-value curve integral on a plane. Not everything translates from the real case as nicely. For example, in the HW5 it is shown that the Mean Value Theorem for integral breaks in the complex case.

Lecture 6. Cauchy Integral Theorem. Some corollaries

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Relevant Sections in Markushevich:
I.13.63–67.

6.1. **Cauchy Integral Theorem.** From this point on, we are going to usually deal with functions differentiable on a "thick" set. To this end, we give the following definitions.

Functions, complex differentiable on an domain D, are called *holomorphic, or analytic* on D. We say that the function f is analytic at a point z_0 if f is analytic on a neighborhood of z_0.

Also, recall that functions that are complex differentiable on the whole complex plane $\mathbb{C}$ are called *entire*.

Theorem 15. (Cauchy Integral Theorem, Cauchy–Goursat Theorem, Cauchy Theorem) *Let G be simply connected domain, and let $f(z)$ be an analytic function on G. Then*

$$\int_L f(z)dz = 0,$$

where L is any closed rectifiable curve contained in G.

Note that here we say that domain G, bounded or not, is simply connected if and only if together with any closed simple curve γ, it also contains its interior $I(\gamma)$.

Proof of Cauchy's integral theorem is organized into 6 steps, each successive step dealing with more complicated curves L.

Step 1. In this step we prove the theorem for L that is a "bigon", that is a curve γ, traversed back and forth: $L = \gamma - \gamma$. In this case statement just follows from additivity of integral over curves.

6.2. **Goursat's Lemma.** Here we deal with a crucial special case of Cauchy Theorem.

Step 2. In this step we prove the theorem for L that is a triangle (three points on plane connected with straight line segments). This step is the essence of the proof of Cauchy Theorem.

Lemma 2. (Goursat's Lemma) *Let G be simply connected domain, and let $f(z)$ be an analytic function on G. Then*

$$\int_L f(z)dz = 0,$$

where L is any triangle contained in G.

To prove this lemma, assume $|\int_L f(z)dz| = M$. Join midpoints of sides of L with straight line segments. This splits $L = L_0$ into four triangles $L^I, L^{II}, L^{III}, L^{IV}$ similar to L with coefficient $1/2$.

Note that

$$\int_L f(z)dz = \int_{L^I} f(z)dz + \int_{L^{II}} f(z)dz + \int_{L^{III}} f(z)dz + \int_{L^{IV}} f(z)dz,$$

where $L^I, L^{II}, L^{III}, L^{IV}$ are traversed in the same directions as L. Therefore for one of these four triangles, which we denote L_1, we have

$$\left|\int_{L_1} f(z)dz\right| \geq \frac{M}{4}.$$

(Otherwise absolute value of sum $\left|\left(\int_{L^I} + \int_{L^{II}} + \int_{L^{III}} + \int_{L^{IV}}\right) f(z)dz\right|$ is less than $4 \cdot \frac{M}{4} = M$ by triangle inequality.)

Doing the same with L_1, obtain triangle L_2. Proceeding in this fashion, we get a sequence of triangles L_n, $n = 1, 2, \ldots$ with the following properties:

(1) Each L_k and its interior $I(L_k)$ is contained in $L_{k-1} \subseteq G$.
(2) $\left|\int_{L_k} f(z)dz\right| \geq \frac{M}{4^k}$.
(3) Perimeter of each triangle L_k is $\frac{l}{2^k}$, where l is the perimeter of $L = L_0$.

Therefore, we have a nested system of closed triangles whose sides $\to 0$ as $n \to \infty$. It follows that they have a common point ζ: $\zeta \in L_n \cup I(L_n)$, $n = 1, 2, \ldots$. Since f is analytic on G, $f'(\zeta)$ exists, that is for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(\zeta) - f'(\zeta)(z-\zeta)| < \varepsilon|z-\zeta|,$$

if $|z - \zeta| < \delta$ (see definition 8). Then, for any L_n contained in the disc $B_\delta(\zeta)$, we have

$$\begin{aligned} & \left|\int_{L_n}(f(z) - f(\zeta) - f'(\zeta)(z-\zeta))dz\right| = \\ = & \left|\int_{L_n} f(z)dz - \int_{L_n} f(\zeta)dz - \int_{L_n} f'(\zeta)(z-\zeta)dz\right| = \\ = & \left|\int_{L_n} f(z)dz - f(\zeta)\int_{L_n} dz - f'(\zeta)\int_{L_n}(z-\zeta)dz\right| = \\ = & \left|\int_{L_n} f(z)dz - 0 - 0\right| = \\ = & \left|\int_{L_n} f(z)dz\right| \end{aligned}$$

Here we used that $\int_\gamma dz = 0$ and $\int_\gamma zdz = 0$ for any closed rectifiable curve γ, as was mentioned above.

On the other hand, for z a point on L_n, we have $|z - \zeta| < \{\text{perimeter of } L_n\} = l/2^n$, so by boundedness property of integral,

$$\left|\int_{L_n} f(z) - f(\zeta) - f'(\zeta)(z-\zeta)\right| < \int_{L_n} \varepsilon|z-\zeta|dz < \varepsilon\frac{l}{2^n}\cdot\frac{l}{2^n} = \frac{\varepsilon l^2}{4^n}.$$

Therefore, comparing the inequality above to the property (2) of L_n, we have

$$\frac{M}{4^n} \le \left|\int_{L_n} f(z)dz\right| \le \frac{\varepsilon l^2}{4^n},$$

so

$$M \le \varepsilon l^2$$

for any $\varepsilon > 0$, so $M = 0$. This finishes proof of Goursat's Lemma.

6.3. **Finishing proof of Cauchy's Integral Theorem.** Now that we have Goursat's Lemma, the rest of the proof is technical.

Step 3. In this step we prove the theorem for L that is a convex n-gon. Since L is a convex polygon, drawing all diagonals from one a vertex of L subdivides L into $n - 2$ triangles. Integral along each triangle is 0 by the Step 2, so sum of integrals along these triangles is also zero. But this sum is equal to the integral along L.

Step 4. In this step we prove the theorem for L that is an arbitrary n-gon. Extending all sides of L to straight lines, we partition L into convex polygons, reducing this case to Step 3. (Alternatively, one can also argue that every n-gon, convex or not, has at least one interior diagonal, and proceed as in Step 3.)

Step 5. In this step we prove the theorem for L that is an arbitrary (possibly, self-intersecting) polygonal curve. In this case we travel along L until we meet the first point of intersecting with path that we already traversed. Note that the loop that we traversed is a polygon (otherwise, we would have met a point of self-intersection earlier). We can discard this loop from L, because by Step 4 integral along it is zero. The remaining part of L has fewer vertices, so proceeding in the same fashion, at some point we reduce L to a non self-intersecting polygonal curve, i.e. to a curve covered by Step 4.

Step 6. Here we prove the theorem for L that is an arbitrary rectifiable curve. This step is technically involved, but at the same time completely standard and not specific to complex analysis. The idea is that we can approximate an arbitrary rectifiable curve by a polygonal curve without changing the value of the integral too much. (Notice that the key statement below, Lemma 4 does not even ask the function to be differentiable, nor the curve to be closed.)

Before we state the corresponding lemma, recall notion of distance between sets. (The following definitions work for arbitrary metric spaces, but right now we only care about $\mathbb{C}$.)

If $z \in \mathbb{C}$, and $A \subseteq \mathbb{C}$, we say that the distance $\operatorname{dist}(z, A)$ is defined by

$$\operatorname{dist}(z, A) = \inf\{|z - w| : w \in A\}.$$

Similarly, if A, B are two subsets of $\mathbb{C}$, we define

$$\operatorname{dist}(A, B) = \inf\{|z - w| : z \in A, w \in B\}.$$

Note that $|z - w|$ is a continuous function of w, so it reaches its minimum and maximum as w runs through points of a set A if A is closed and bounded. Moreover, $|z - w|$ is nonnegative and is $\geq R$ outside the circle $|z - w| < R$ centered at z. Therefore, $\operatorname{dist}(z, A)$ is attained as $|z - w|$ for some $w \in A$ if A is closed.

Further, it is not hard to show that $\operatorname{dist}(z, A)$ is a continuous function of z, so if A, B are closed and A is bounded, then $\operatorname{dist}(A, B)$ is attained as $|z - w|$ for some $z \in A, w \in B$. Indeed, $\operatorname{dist}(A, B)$ is attained as some $\operatorname{dist}(z, B)$ since A is closed and bouned, and $\operatorname{dist}(z, B)$ is attained as $|z - w|$ since B is closed. One important corollary is the following.

Lemma 3. *If A, B are closed and one of them bounded, then $A \cap B = \emptyset$ if and only if $\operatorname{dist}(A, B) > 0$.*

Note that if both are unbounded, this may not be true, e.g. A the hyperbola $y = 1/x$ and B the x-axis.

Finally, a bit of notation: for a set A, by $\mathcal{N}_r(A)$ we denote its closed r-neighborhood:

$$\mathcal{N}_r(A) = \{z \in \mathbb{C} \mid \operatorname{dist}(z, A) \leq r\}.$$

Since $\operatorname{dist}(z, A)$ is a continuous function of z, a closed neighborhood is a closed set. (Exercise: prove that).

Reminder: for curves γ_1 and γ_2 s.t. the endpoint of γ_1 is the start point of γ_2, by $\gamma_1 + \gamma_2$ we denote the concatenation of these curves, i.e. the curve that traverses γ_1, and then γ_2. Also, by $\overrightarrow{z_1 z_2}$ we denote the straight line segment connecting z_1 to z_2.

Lemma 4. *Let $f(z)$ be a continuous function on a domain G, L be an arbitrary rectifiable curve on G given by $z = \lambda(t)$, $\lambda : [a, b] \to G$. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that for any partition $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with $|\mathcal{P}| < \delta$, the polygonal curve $\Lambda = \overrightarrow{z_0 z_1} + \overrightarrow{z_1 z_2} + \ldots + \overrightarrow{z_{n-1} z_n}$ where $z_k = \lambda(t_k)$, $k = 0, 1, \ldots, n$, is contained in G, and*

$$\left| \int_L f(z) dz - \int_\Lambda f(z) dz \right| < \varepsilon.$$

Proof. First, we find δ that delivers first part of statement, that is that $\Lambda \subseteq G$. Since λ is a continuous function on closed interval $[a, b]$, it is uniformly continuous on this interval. Let $d = \operatorname{dist}(L, \mathbb{C} \setminus G)$. By the Lemma 3 above, $d > 0$. Put $\delta' > 0$ to be such that $|\lambda(t) - \lambda(t')| < d$ whenever $|t - t'| < \delta'$.

Then for any partition $\dot{\mathcal{P}}$ with $|\dot{\mathcal{P}}| < \delta'$, note that any point on the polygonal curve $\Lambda = \overrightarrow{z_0 z_1} + \ldots + \overrightarrow{z_{n-1} z_n}$ lies distance at most $d/2$ from the corresponding endpoint z_k, i.e. $\Lambda \subseteq \mathcal{N}_{d/2}(L) \subseteq G$.

We established that there exists $\delta' > 0$ such that $\Lambda \subseteq G$ whenever $|\mathcal{P}| < \delta'$.

Now, we need to find $\delta_0 \leq \delta'$ that provides $\left|\int_L f(z)dz - \int_\Lambda f(z)dz\right| < \varepsilon$. Since f is a function continuous on a closed bounded set $\mathcal{N}_{d/2}(L)$, f is uniformly continuous on the same set, that is for any $\varepsilon > 0$, there is a $\delta'' > 0$ such that $|f(z) - f(z')| < \frac{\varepsilon}{2l}$, whenever $|z - z'| < \delta''$, where l denotes length of L.

By definition of integral, there is a $\delta_1 > 0$ such that

$$\left|\int_L f(z)dz - \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1})\right| < \varepsilon/2 \tag{10}$$

for any partition $\mathcal{P}$ with $|\mathcal{P}| < \delta_1$.

Take $\delta_0 > 0$ such that $\delta_0 < \delta'$, $\delta_0 < \delta_1$, and $|\lambda(t) - \lambda(t')| < \delta''$ if $|t - t'| < \delta_0$. For any partition $\mathcal{P}$ with $|\mathcal{P}| < \delta_0$ denote segment $\overrightarrow{z_{k-1} z_k}$ by Λ_k. Then

$$\left|\int_\Lambda f(z)dz - \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1})\right| =$$

$$= \left|\sum_{k=1}^{n} \int_{\Lambda_k} (f(z) - f(z_k))dz\right| \leq \frac{\varepsilon}{2l} \cdot l = \varepsilon/2,$$

i.e.

$$\left|\int_\Lambda f(z)dz - \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1})\right| \leq \varepsilon/2 \tag{11}$$

Then for any partition $\mathcal{P}$ with $|\mathcal{P}| < \delta_0$, we have that $\Lambda \subseteq G$, and comparing (10) and (11), by triangle inequality we get

$$\left|\int_L f(z)dz - \int_\Lambda f(z)dz\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This finishes proof of lemma. □

Now, back to proof of Step 6. For an arbitrary closed rectifiable curve L in G, we take arbitrary $\varepsilon > 0$ and arbitrary approximation of L by a polygonal curve Λ induced by partition as in statement of the above Lemma. Then we get

$$\left|\int_L f(z)dz - \int_\Lambda f(z)dz\right| < \varepsilon,$$

but by Step 5 we know that $\int_\Lambda f(z)dz = 0$, so

$$\left|\int_L f(z)dz\right| < \varepsilon,$$

for any ε, so $\left|\int_L f(z)dz\right| = 0$. This finishes proof of Cauchy's Integral Theorem.

6.4. **Stronger version of Cauchy Integral Theorem.**

Theorem 16. *Let G be the interior of a closed simple curve L, $G = I(L)$. Let f be analytic on G and continuous on $\overline{G} = G \cup L$. Then*

$$\int_L f(z)dz = 0.$$

Proof of this theorem is skipped (the proof is accessible at this point in the case of G which is "nice" in certain way, *star shaped*. In general case, the proof requires a whole new idea, approximation of functions by polynomials, and does not fit in this course.)

Note that this version of Cauchy Theorem allows to deal with the following integral, while original version does not:

$$\int_L \sqrt{z}dz = 0,$$

where L is a circle $|z-1| = 1$ passing through 0, and value of $\sqrt{z}$ is chosen so that resulting function is continuous on $|z-1| \leq 1$. Note that $\sqrt{z}$ cannot be defined as a single-valued analytic function on a neighborhood of 0, so the original version of Cauchy theorem does not apply.

We will not rely on this theorem in the course.

6.5. **Cauchy Theorem for a system of contours.** Let's recall how exactly we used simple connectedness of G: we actually only needed that the interior $I(L)$ is contained in G. That allows us to state the following theorem:

Theorem 17. *Let G be an arbitrary domain and let $f(z)$ be an analytic function on G. Then*

$$\int_L f(z)dz = 0,$$

where L is any closed simple rectifiable curve such that G contains both L and $I(L)$.

This simple reformulation allows us to prove the following fact.

Theorem 18. (Cauchy's Integral Theorem for a system of contours) *Let G be an arbitrary domain, let $f(z)$ be analytic on G. Let $\Gamma, \gamma_1, \ldots, \gamma_n$ be a system of $n+1$ simple closed rectifiable curves in G s.t.*

(1) $\gamma_1, \ldots, \gamma_n \subseteq I(\Gamma)$,
(2) $\gamma_j \subseteq E(\gamma_k)$ *for any $j \neq k$.,*
(3) *G contains $D = I(\Gamma) \setminus \left(\overline{I(\gamma_1)} \cup \ldots \cup \overline{I(\gamma_n)}\right)$.*

Then

$$\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz,$$

where all integrals are traversed in the same direction.

Proof. (A figure will be provided in later drafts.) Pick points A_0 on Γ, A_1, B_1 on γ_1, ..., A_n, B_n on γ_n, B_0 on Γ. Let γ_k' be an arc of γ_k from B_k to A_k clockwise, and γ_k'' be an arc of γ_k from A_k to B_k clockwise. Let δ_k be a simple polygonal path in G from A_{k-1} to B_k, $k = 1, \ldots, n$, and δ_{n+1} be a simple polygonal path in G from A_n to B_0. Let Γ' be an arc of Γ from B_0 to A_0 counterclockwise and Γ'' be an arc of Γ from A_0 to B_0 counterclockwise. Then for closed contours

$$L' = \delta_1 + \gamma_1' + \delta_2 + \gamma_2' + \ldots + \gamma_n' + \delta_{n+1} + \Gamma'$$

and

$$L'' = -\delta_{n+1} + \gamma_1'' - \delta_2 + \gamma_2'' - \ldots + \gamma_n'' - \delta_{n+1} + \Gamma''$$

by the previous theorem we have

$$\int_{L'} f(z)dz = \int_{L''} f(z)dz = 0,$$

but

$$0 = \int_{L'+L''} f(z)dz = \int_\Gamma f(z)dz + \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz,$$

where Γ is traversed counterclockwise and γ_k are traversed clockwise, as required. □

6.6. **Application of Cauchy Integral Theorem to real variable integrals.** Cauchy Integral Theorem and its multiple contour version can be used nicely to compute real variable integrals.

6.6.1. *Integral of a rational function over* $\mathbb{R}$. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Computing this integral using standard real calculus techniques is a simple task. Nevertheless, here we provide a way to compute this integral using Cauchy's Integral Theorem. It will be clear that similar approach works for any rational function for which the integral over the real line converges.

Consider the function $f(z) = \frac{1}{z^2+1}$. This function is differentiable everywhere on $\mathbb{C}$, save for $z = \pm i$. Consider two contours, L_1 that consists of the segment AB: $-R \le x \le R$, of the real line, and upper semicircle BCA of radius R centered at the origin; and L_2 which is a circle $|z - i| = \varepsilon$, $\varepsilon < 1$. Also let $R > 1 + \varepsilon$.

By Cauchy's Integral Theorem for a system of contours, $\int_{L_1} \frac{dz}{z^2+1} = \int_{L_2} \frac{dz}{z^2+1}$. First we treat the integral along L_1.

On the segment AB we have $z = x$, $dz = dx$, therefore

$$J_1(R) = \int_{AB} \frac{dz}{z^2+1} = \int_{-R}^{R} \frac{dx}{x^2+1},$$

so $J_1(R) \longrightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ as $R \to \infty$ if the latter improper integral converges.

On the semicircle BCA we have $z = Re^{i\theta}$, $0 \le \theta \le \pi$, $dz = iRe^{i\theta}d\theta$, so

$$J_2(R) = \int_{BCA} \frac{dz}{z^2+1} = \int_0^\pi \frac{iRe^{i\theta}d\theta}{R^2e^{2i\theta}+1},$$

so if $R \ge 2$,

$$|J_2(R)| \le R\int_0^\pi \frac{d\theta}{|R^2e^{2i\theta}+1|} \le R\int_0^\pi \frac{d\theta}{R^2/2} = \frac{2\pi}{R},$$

hence $J_2(R) \to 0$ as $R \to \infty$. Finally,

$$J_3(\varepsilon) = \int_{L_2} \frac{dz}{z^2+1} = \int_{L_2} \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz =$$

$$= \int_{L_2} \frac{1}{2i}\frac{dz}{z-i} - \int_{L_2} \frac{1}{2i}\frac{dz}{z+i} = \frac{1}{2i}2\pi i - 0 = \pi,$$

since $1/(z+i)$ is analytic inside L_2. Taking limit as $R \to \infty$ of the equality

$$J_1(R) + J_2(R) = J_3,$$

we get
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi.$$

We will start the next lecture with looking at more applications of Cauchy Integral Theorem to real variable integrals.

6.6.2. *Integral Sine at infinity.* Evaluate the integral
$$\int_0^{\infty} \frac{\sin x}{x} dx.$$

COMMENT. The function $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ is called the *integral sine* function and, along with $\operatorname{sinc} x = \frac{\sin x}{x}$, is used in signal processing and Fourier analysis in general.

Consider the function
$$f(z) = \frac{e^{iz}}{z},$$
differentiable everywhere except $z = 0$, integrated along the path L which is an "upper semiring" of inner radius r and outer radius R, as shown in Figure 2.

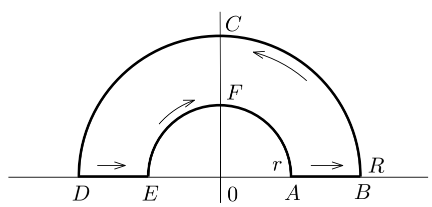

FIGURE 2. Path L.

By Cauchy Theorem, we have
$$0 = \int_L \frac{e^{iz}}{z} dz = \int_{AB} \frac{e^{iz}}{z} dz + \int_{BCD} \frac{e^{iz}}{z} dz + \int_{DE} \frac{e^{iz}}{z} dz + \int_{EFA} \frac{e^{iz}}{z} dz.$$

Compute/estimate these integrals separately. On AB, we have $z = t$, where t runs from r to R, so $dz = dt$ and we have
$$J_1 = \int_{AB} \frac{e^{iz}}{z} dz = \int_r^R \frac{e^{it}}{t} dt = \int_r^R \frac{\cos t}{t} dt + i \int_r^R \frac{\sin t}{t} dt. \tag{12}$$

On the circular arc BCD we have $z = Re^{it}$, where t runs from 0 to π, so $dz = iRe^{it}dt$, and
$$J_2 = \int_{BCD} \frac{e^{iz}}{z} dz = \int_0^{\pi} \frac{\exp(iRe^{it})}{Re^{it}} iRe^{it} dt = i \int_0^{\pi} \exp(iR\cos t - R\sin t) dt.$$
Therefore,
$$|J_2| \le \int_0^{\pi} |\exp(iR\cos t - R\sin t)| dt = \int_0^{\pi} \exp(-R\sin t) dt = 2\int_0^{\pi/2} \exp(-R\sin t) dt.$$
Note that on $[0, \pi/2]$, $\sin t \ge 2t/\pi$ (look at the graphs of $\sin t$ and of $2t/\pi$), so
$$|J_2| \le 2\int_0^{\pi/2} \exp(-R\sin t) dt \le 2\int_0^{\pi/2} \exp(-2Rt/\pi) dt =$$

$$= 2\,\frac{\exp(-2Rt/\pi)}{-2R/\pi}\bigg|_0^{\pi/2} = \pi\frac{1-e^{-R}}{R} < \frac{\pi}{R}.$$

Therefore,

$$J_2 \to 0 \text{ as } R \to \infty. \tag{13}$$

On DE, we have $z = t$, where t runs from $-R$ to $-r$, so $dz = dt$, and similarly to J_1,

$$J_3 = \int_{DE} \frac{e^{iz}}{z}dz = \int_{-R}^{-r} \frac{\cos t}{t}dt + i\int_{-R}^{-r} \frac{\sin t}{t}dt = -\int_r^R \frac{\cos t}{t}dt + i\int_r^R \frac{\sin t}{t}dt.$$

Putting this together with (12), we get that

$$J_1 + J_3 = 2i\int_r^R \frac{\sin t}{t}dt. \tag{14}$$

Finally, on EFA, we have $z = re^{it}$, where t runs from π to 0, so $dz = ire^{it}$ and

$$J_4 = \int_{EFA} \frac{e^{iz}}{z}dz = \int_\pi^0 \frac{\exp(ire^{it})}{re^{it}}ire^{it}dt = -i\int_0^\pi \exp(ir\cos t - r\sin t)dt.$$

Since e^{iz} is continuous at 0, given $\varepsilon > 0$, find r small enough to guarantee that $|e^{iz} - 1| < \varepsilon$ on EFA. Then

$$\left|J_4 - (-i\int_0^\pi 1dt)\right| = \left|-i\int_0^\pi (\exp(ir\cos t - r\sin t) - 1)dt\right| \le \left|\int_0^\pi \varepsilon dt\right| = \pi\varepsilon,$$

so

$$\lim_{r\to 0} J_4 = -i\int_0^\pi 1dt = -i\pi. \tag{15}$$

Recall that by definition,

$$\int_0^\infty \frac{\sin t}{t}dt = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{\sin t}{t}dt.$$

Now, we look at the limit of the equality

$$0 = J_1 + J_2 + J_3 + J_4$$

as $R \to \infty$ and $r \to 0$. By (13),(14),(15), we get

$$\begin{array}{ccccccc} 0 & = & (J_1 + J_3) & + & J_2 & + & J_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & = & 2i\displaystyle\int_0^\infty \frac{\sin t}{t}dt & + & 0 & + & (-i\pi), \end{array}$$

that is,

$$0 = 2i\int_0^\infty \frac{\sin t}{t}dt + (-\pi i).$$

so

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$$

Remark. Computing this integral using real analysis methods is rather involved. For example, note that this integral absolutely diverges, so powerful tools of Lebesgue integration like the Lebesgue dominated convergence theorem do not apply.

6.6.3. *Fresnel Integrals (optional section, was not included in the lecture).* Evaluate

$$\int_0^\infty \cos x^2 dx, \quad \int_0^\infty \sin x^2 dx.$$

COMMENT. These are called Fresnel integrals. Functions $C(x) = \int_0^x \cos t^2 dt$ and $S(x) = \int_0^x \sin t^2 dt$ arise in several areas of geometry, physics (like theory of diffractions), and engineering.

We will use two auxiliary facts. First one is the formula

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

The second one is the inequality

$$\sin\theta \geq \frac{\pi\theta}{2} \quad \left(0 \leq \theta \leq \frac{\pi}{2}\right).$$

To evaluate the Fresnel integrals, we consider the function e^{iz^2}. Since e^z and iz^2 are differentiable everywhere on $\mathbb{C}$, by chain rule e^{iz^2} is also differentiable everywhere on $\mathbb{C}$.

Consider closed contour L consisting of the segment OA: $0 \leq x \leq R$ of the nonnegative real axis, the $\frac{\pi}{4}$ arc AB of the circle of radius R centered at origin, and the segment BO of the line bisecting the angle formed by nonnegative real and imaginary axes. By Cauchy's Integral Theorem,

$$\int_L e^{iz^2} dz = \int_{OA} e^{iz^2} dz + \int_{AB} e^{iz^2} dz + \int_{BO} e^{iz^2} dz = 0.$$

Compute these three integrals separately. Segment OA is parameterized by a real x, so

$$J_1(R) = \int_{OA} e^{iz^2} dz = \int_0^R e^{ix^2} dx = \int_0^R \cos x^2 dx + i\int_0^R \sin x^2 dx.$$

On the arc AB, $z = Re^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$, $z^2 = R^2 e^{2i\theta}$, $dz = iRe^{i\theta} d\theta$, so

$$J_2(R) = \int_{AB} e^{iz^2} dz = \int_0^{\pi/4} \exp(iR^2 e^{2i\theta}) iRe^{i\theta} d\theta.$$

Finally, on the segment BO we have

$$z = re^{i\pi/4},\ R \geq r \geq 0,\ z^2 = r^2 e^{i\pi/2} = ir^2,\ dz = e^{i\pi/4} dr$$

and

$$J_3(R) = \int_{BO} e^{iz^2} dz = \int_R^0 e^{-r^2} e^{i\pi/4} dr = -e^{i\pi/4} \int_0^R e^{-r^2} dr =$$
$$= -\frac{\sqrt{2}}{2}(1+i)\int_0^R e^{-r^2} dr.$$

Now, let R approach infinity: $R \to \infty$. Then

$$J_3(R) = -\frac{\sqrt{2}}{2}(1+i)\int_0^R e^{-r^2} dr \longrightarrow -\frac{\sqrt{2}}{2}(1+i)\int_0^\infty e^{-r^2} dr =$$
$$= -\frac{\sqrt{2}}{2}(1+i)\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4}(1+i).$$

Show that $J_2(R) \to 0$ as $R \to \infty$.

$$|J_2(R)| \le R \int_0^{\pi/4} |\exp(iR^2 e^{2i\theta})| d\theta.$$

Note that $|\exp(iR^2 e^{2i\theta})| = \exp(-R^2 \sin 2\theta)$, so

$$|J_2(R)| \le R \int_0^{\pi/4} \exp(-R^2 \sin 2\theta) d\theta.$$

Recall that $\sin 2\theta \ge 4\theta/\pi$ if $0 \le 2\theta \le \pi/2$. Therefore,

$$|J_2(R)| \le R \int_0^{\pi/4} \exp(-R^2 \cdot 4\theta/\pi) d\theta = R \frac{\exp(-4R^2\theta/\pi)}{-4R^2/\pi}\Big|_{\theta=0}^{\theta=\pi/4} =$$

$$= \frac{\pi}{4} \frac{1-e^{-R^2}}{R} \le \frac{\pi}{4R},$$

so $\lim_{R\to\infty} J_2(R) = 0$, as claimed. Finally, consider $J_1(R)$. Since $J_1(R) + J_2(R) + J_3(R) = 0$, it follows that

$$\lim_{R\to\infty} J_1(R) = -\lim_{R\to\infty} J_2(R) - \lim_{R\to\infty} J_3(R) = \frac{\sqrt{2\pi}}{4}(1+i),$$

i.e.

$$\lim_{R\to\infty} \int_0^R \cos x^2 dx + i \int_0^R \sin x^2 dx = \frac{\sqrt{2\pi}}{4}(1+i),$$

which means that improper integrals

$$\int_0^\infty \cos x^2 dx = \lim_{R\to\infty} \int_0^R \cos x^2 dx$$

and

$$\int_0^\infty \sin x^2 dx = \lim_{R\to\infty} \int_0^R \sin x^2 dx$$

exist and

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

Remark. Computing these integrals using tools of real variable analysis is also rather involved.

Lecture 7. Antiderivative. Cauchy's Integral Formula and its Corollaries

March 8, 2017
Relevant Sections in Markushevich:
I.13.68, I.14.70–72.

7.1. **Antiderivative. Fundamental theorem of calculus.**

Definition 14. *Let $f(z)$ be a function on a domain G. A function $F(z)$ defined and analytic on G is called an* antiderivative (primitive, indefinite integral) *of $f(z)$ if $F'(z) = f(z)$ at each point $z \in G$.*

For a function f analytic on simply connected domain G, integral along a closed rectifiable curve vanishes. Therefore, the integral

$$\int_{z_0}^{z} f(\zeta)d\zeta$$

does not depend on a particular choice of path between two points z_0, z, so for a fixed z_0, this integral is a well-defined function of z:

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta.$$

Theorem 19. *Let G be a simply connected domain, let $f(z)$ be differentiable on G. Then $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ is differentiable on G and $F'(z) = f(z)$ for each $z \in G$.*

Proof. Compute $F'(z)$ by definition.

$$\Delta F = \int_{z_0}^{z+\Delta z} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta = \int_{z}^{z+\Delta z} f(\zeta)d\zeta =$$

$$= \Delta z f(z) + \int_{z}^{z+\Delta z} (f(\zeta) - f(z))d\zeta.$$

Then

$$\frac{\Delta F}{\Delta z} = f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} (f(\zeta) - f(z))d\zeta,$$

so

$$\begin{array}{rcl} \left|\frac{\Delta F}{\Delta z} - f(z)\right| & = & \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z}(f(\zeta) - f(z))d\zeta\right| \leq \\ & \leq & \left|\frac{1}{\Delta z}M(r)\cdot\Delta z\right| = M(r) \to 0 \end{array}$$

as $\Delta z \to 0$ since f is continuous. Here $M(r)$ denotes $\max|f(z) - f(\zeta)|$ as ζ runs through values such that $|z - \zeta| \leq r$.

By definition of derivative, $F'(z) = f(z)$. □

Remark. Note that we did not use differentiability of f other than to get path independence of F. If we somehow know that integral representing F is path independent, it is enough to require that f is merely continuous.

Theorem 20. *Let G be a simply connected domain, and let $f(z)$ be a differentiable function on G. Then any antiderivative of $f(z)$ on G can be represented as*

$$\Phi(z) = \int_{z_0}^{z} f(\zeta)d\zeta + C,$$

where $z_0 \in G$ and C is a constant.

Proof. This follows by one of problems in Homework 5. Just in case, we provide the proof here, too.

Write

$$\varphi(z) = \Phi(z) - \int_{z_0}^{z} f(\zeta)d\zeta = u(x,y) + iv(x,y).$$

Then $\varphi'(z) = 0$, so

$$0 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

Therefore, $u(x, y) = const$, $v(x, y) = const$, so $\varphi(z) = const$. □

Theorem 21. (Fundamental theorem of integral calculus for complex functions). *Let G be a simply connected domain, and let $f(z)$ be a differentiable function on G. Then if $z_0, z \in G$,*

$$\int_{z_0}^{z} f(\zeta)d\zeta = \Phi(z) - \Phi(z_0),$$

where $\Phi(z)$ is any antiderivative of $f(z)$ on G.

Proof. By the previous theorem, it suffices to check this statement for $\Phi(z) = \int_{z_0}^{z} f(\zeta)d\zeta$. □

The latter statement allows to compute integrals of differentiable functions just as we are used to, for example:

$$\int_{z_0}^{z} \zeta^k d\zeta = \frac{z^{k+1}}{k+1} - \frac{z_0^{k+1}}{k+1} \qquad \text{if } k \in \mathbb{Z}, k \neq -1,$$

$$\int_{z_0}^{z} e^{\zeta} d\zeta = e^{z} - e^{z_0},$$

$$\int_{z_0}^{z} \cos\zeta d\zeta = \sin z - \sin z_0,$$

$$\int_{z_0}^{z} \sin\zeta d\zeta = \cos z_0 - \cos z.$$

NOTE that the first equality for $k \geq 0$ follows by the Corollary above, but for $k \leq -2$, one needs to apply the Remark after Theorem 19, since z^k in that case is not differentiable at 0, but nevertheless integrals *are* path-independant, as direct computation shows (for example, see Homework 6).

7.2. **Cauchy's Integral Formula.** The theorem we prove below is of fundamental importance in complex analysis.

Theorem 22. (Cauchy's Integral Formula) *If $f(z)$ is differentiable on a domain G, and if G contains a closed simple rectifiable curve γ and its interior $I(\gamma)$, then*

$$\int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z) \quad \text{if } z \in I(\gamma),$$

and

$$\int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = 0 \quad \text{if } z \in E(\gamma),$$

where $E(\gamma)$ denotes exterior of γ.

Note that it is assumed that γ is traversed counterclockwise, that is $I(\gamma)$ is to the left of an observer moving along γ.

Under assumptions of this theorem, the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

is called *Cauchy's integral*.

Proof. Let z be arbitrary in $I(\gamma)$. Then the function

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

is analytic on $G' = G\backslash\{z\}$. Let ρ be so small that the closed disc $\overline{B_\rho(z)}$ is contained in $I(\gamma)$. Then, if γ_ρ denotes circle $|\zeta - z| = \rho$, by Cauchy's Integral Theorem for a system of contours γ, γ_ρ we have

$$\int_\gamma g(\zeta)d\zeta = \int_{\gamma_\rho} g(\zeta)d\zeta,$$

that is

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Since $\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$ does not depend on a particular choice of ρ, neither does $\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta$, so

$$\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta = \lim_{\rho\to 0}\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

So to prove statement of the theorem for $z \in I(\gamma)$, it suffices to prove that

$$\lim_{\rho\to 0}\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta = 2\pi i f(z).$$

That is, we need to prove that for any $\varepsilon > 0$, one can find $\delta > 0$ such that

$$\left|\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta - 2\pi i f(z)\right| < \varepsilon$$

whenever $\rho < \delta$. Write $f(\zeta) = f(z) + \alpha(\zeta)$. Since

$$\int_{\gamma_\rho} \frac{d\zeta}{\zeta - z} = 2\pi i,$$

we see that

$$\left|\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta - 2\pi i f(z)\right| =$$

$$= \left|\int_{\gamma_\rho} \frac{f(z) + \alpha(\zeta)}{\zeta - z}d\zeta - f(z)\int_{\gamma_\rho} \frac{d\zeta}{\zeta - z}\right| =$$

$$= \left|\int_{\gamma_\rho} \frac{\alpha(\zeta)}{\zeta - z}d\zeta\right|.$$

Now, since $f(\zeta)$ is continuous at $\zeta = z$, for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|\alpha(\zeta)| < \frac{\varepsilon}{2\pi}$$

whenever $|z-\zeta|<\delta$. Therefore, if $\rho<\delta$, we have

$$\left|\int_{\gamma_\rho}\frac{f(\zeta)}{\zeta-z}d\zeta-2\pi i f(z)\right|=\left|\int_{\gamma_\rho}\frac{\alpha(\zeta)}{\zeta-z}d\zeta\right|\le\left|\int_{\gamma_\rho}\frac{|\alpha(\zeta)|}{\rho}d\zeta\right|<\frac{\varepsilon/2\pi}{\rho}2\pi\rho=\varepsilon,$$

as desired.

It is only left to note that the case when $z\in E(\gamma)$ is a direct consequence of Cauchy Integral Theorem. □

Example. Straightforward application of Cauchy Integral Formula allows to evaluate certain integrals without much computation. For example, show that

$$\int_\gamma\frac{dz}{z^2+1}=0$$

for any simple closed rectifiable curve γ s.t. $\pm i\in I(\gamma)$. Let γ_1, γ_2 be circles $\subseteq I(\gamma)$ centered at $-i, i$, respectively, and traversed in the same direction as γ. Then we have by Cauchy Theorem for multiple contours (Therem 18) and by Cauchy Integral Formula

$$\begin{aligned}\int_\gamma\frac{dz}{z^2+1} &= \int_{\gamma_1}\frac{dz}{z^2+1}+\int_{\gamma_2}\frac{dz}{z^2+1}=\\ &= \int_{\gamma_1}\frac{1/(z-i)}{z+i}dz+\int_{\gamma_2}\frac{1/(z+i)}{z-i}dz=2\pi i\tfrac{1}{-i-i}+2\pi i\tfrac{1}{i+i}=0.\end{aligned}$$

7.2.1. *Average value of an analytic complex function.* Below is the first corollary to Cauchy Integral Formula.

Theorem 23. *If $f(z)$ is a differentiable function on a domain G, and if G contains the circle γ_ρ: $|z-z_0|=\rho$ and its interior $I(\gamma_\rho)$, then*

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}f(z_0+\rho e^{i\theta})d\theta,$$

i.e., the value of $f(z)$ at z_0 equals to the average of its values on the circle γ_ρ with center z_0.

Proof. The equation of γ_ρ is

$$z=z_0+\rho e^{i\theta},\ 0\le\theta\le 2\pi.$$

By Cauchy's integral formula,

$$f(z_0)=\frac{1}{2\pi i}\int_{\gamma_\rho}\frac{f(z)dz}{z-z_0}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}}i\rho e^{i\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(z_0+\rho e^{i\theta})d\theta.$$

□

Note that in real variable analysis there is a class of functions with the same average-value property, harmonic functions, that is, solutions of Laplace equation $\Delta f=0$, e.g. $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$. We will discuss harmonic functions and their connection to complex analytic functions in more detail later on.

The following is a direct corollary of the above Theorem 23.

Corollary 1. *If $f(z)$ is a differentiable function on a domain G, and if G contains the circle γ_ρ: $|z-z_0|=\rho$ and its interior $I(\gamma_\rho)$, then*

$$|f(z_0)|\le M(\rho)=\max_{z\in\gamma_\rho}|f(z)|.$$

Proof. By Theorem 23,

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{i\theta})d\theta\right| \le \frac{1}{2\pi}M(\rho)\cdot 2\pi = M(\rho).$$

□

Note that since γ_ρ is a closed bounded set and f is continuous, $M(\rho)$ is attained at some point $\zeta \in \gamma_\rho$: $M(\rho) = |f(\zeta)|$.

Theorem 24. *If f is an differentiable on a domain G, then $|f(z)|$ cannot have a local strict maximum at any point of G.*

Proof. Suppose z_0 is a point of local strict maximum of $|f(z)|$. Let γ_ρ be same as above. Then $|f(z_0)| \le M(\rho) = |f(\zeta)|$. Since ρ can be chosen arbitrarily small, in any neighborhood of z_0 there is a point ζ such that $|f(z_0)| \le |f(\zeta)|$, so the inequality $|f(z_0)| > |f(\zeta)|$ does not hold and, therefore, z_0 is not a point of local strict maximum. □

Later on we will strengthen this theorem by showing that it also holds for non-strict maximum (with obvious exception to constant functions).

7.3. **Integrals of the Cauchy type.** By an *integral of the Cauchy type* we mean expression of the form

$$\frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)}{\zeta - z}d\zeta,$$

where L is a rectifiable curve (not necessarily closed), $z \notin L$, φ is a function (sometimes called *density*) continuous on L.

Theorem 25. *Every integral of the Cauchy type*

$$f(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)}{\zeta - z}d\zeta$$

defines an infinitely differentiable function $f(z)$ on any domain G containing no points of L. Moreover,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_L \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}d\zeta \quad (n = 0, 1, 2, \ldots).$$

Proof. We prove this statement by induction on n. For $n = 0$, the corresponding formula is

$$f^{(0)}(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)}{(\zeta - z)^1}d\zeta,$$

which is the definition of $f(z)$. Suppose we established that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_L \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for a specific n. Find $f^{(n+1)}$ by straightforward differentiation of $f^{(n)}$:

$$f^{(n+1)}(z_0) = \lim_{z\to z_0} \frac{f^{(n)}(z) - f^{(n)}(z_0)}{z - z_0}$$

where z_0 is arbitrary point in G. For ρ small enough, G contains circle $\gamma_\rho = |z-z_0| = \rho$ and its interior. Let δ be distance between γ_ρ and the curve L. Moreover, let R be large enough to contain L and γ_ρ. Then

$$f^{(n)}(z) - f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_L \varphi(\zeta) \frac{(\zeta - z_0)^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z_0)^{n+1}(\zeta - z)^{n+1}} d\zeta.$$

Write $t = \zeta - z_0$, $h = z - z_0$, so $\zeta - z = t - h$. Then using identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$

we get

$$\frac{f^{(n)}(z) - f^{(n)}(z_0)}{h} = \frac{n!}{2\pi i} \int_L \varphi(\zeta) \frac{(t-h)^n + t(t-h)^{n-1} + \ldots + t^n}{t^{n+1}(t-h)^{n+1}} d\zeta.$$

Since our goal is to show that the expression above approaches the limit

$$\psi(z_0) = \frac{(n+1)!}{2\pi i} \int_L \frac{\varphi(\zeta)}{(\zeta - z_0)^{n+2}} d\zeta = \frac{(n+1)!}{2\pi i} \int_L \frac{\varphi(\zeta)}{t^{n+2}} d\zeta,$$

as $h \to 0$, we now examine the difference (the point of the computation below is that the terms t^{n+1}, i.e., the terms without h, cancel out, as you can see by inspecting numerator in the second line of the computation)

$$\frac{f^{(n)}(z) - f^{(n)}(z_0)}{h} - \psi(z_0) =$$

$$= \frac{n!}{2\pi i} \int_L \varphi(\zeta) \frac{t(t-h)^n + t^2(t-h)^{n-1} + \ldots + t^{n+1} - (n+1)(t-h)^{n+1}}{t^{n+2}(t-h)^{n+1}} d\zeta =$$

$$= \frac{n!}{2\pi i} \int_L \varphi(\zeta) \frac{h(t-h)^n + h[t + (t-h)](t-h)^{n-1} + \ldots + h[t^n + \ldots + (t-h)^n]}{t^{n+2}(t-h)^{n+1}} d\zeta =$$

$$= \frac{n! \cdot h}{2\pi i} \int_L \varphi(\zeta) \frac{(t-h)^n + [t + (t-h)](t-h)^{n-1} + \ldots + [t^n + \ldots + (t-h)^n]}{t^{n+2}(t-h)^{n+1}} d\zeta.$$

Recall that $0 < \delta \leq |\zeta - z_0| = |t| \leq 2R$, and $0 < \delta \leq |\zeta - z| = |t - h| \leq 2R$. Therefore

$$\left| \frac{f^{(n)}(z) - f^{(n)}(z_0)}{h} - \psi(z_0) \right| \leq$$

$$\leq \frac{n! \cdot |h|}{2\pi} \cdot M \cdot \frac{(2R)^n + 2(2R)^{n-1} + \ldots + n(2R)^n}{\delta^{2n+3}} \cdot l,$$

where l is the length of L and

$$M = \max_{\zeta \in L} |\varphi(\zeta)|.$$

The righthand side goes to 0 as $h \to 0$, so

$$f^{(n+1)}(z_0) = \lim_{z \to z_0} \frac{f^{(n)}(z) - f^{(n)}(z_0)}{z - z_0} = \frac{(n+1)!}{2\pi i} \int_L \frac{\varphi(\zeta)}{(\zeta - z_0)^{n+2}} d\zeta.$$

□

7.4. **Infinite differentiability of complex differentiable functions; Morera's Theorem.** An important corollary (below) is that complex differentiable functions are *infinitely* differentiable. Note that in general, this is not the case *at all* when dealing with real-differentiable functions. For instance, $f(x) = x^2 \sin(1/x^3)$ extended to $x = 0$ by 0 is differentiable everywhere on $\mathbb{R}$ but its derivative is not bounded on a neighborhood of 0, therefore it's discontinuous at 0. A less exotic function, $f(x) = x|x|$ is an example of an everywhere differentiable function whose derivative is continuous but not differentiable everywhere. Worse, if $f : \mathbb{R} \to \mathbb{R}$ is a function continuous everywhere but not differentiable everywhere (Weierstrass gave an example of such a function; one can also prove that they exists by a Baire category argument), then its integral $\int_{t_0}^{t} f(\tau)d\tau$ is differentiable at every point but not differentiable twice *at any point.*

Corollary 2. *If $f(z)$ is a differentiable function on a domain G, then $f(z)$ is infinitely differentiable on G.*

Proof. Let $z_0 \in G$. Choose ρ small enough so the circle $\gamma_\rho : |z-z_0| = \rho$ is contained in G together with its interior $I(\gamma_\rho)$. Then by Cauchy's Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z_0} d\zeta,$$

so f is represented by an integral of the Cauchy type. By Theorem 25, f is infinitely differentiable. □

Note that, in particular, we have an expression for n-th derivative:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

where γ_ρ can (by Cauchy's integral theorem) be replaced by any closed rectifiable simple curve L contained in G together with its interior such that $z \in I(L)$.

Below is the same result in a slightly different wording.

Corollary 3. *If $f(z)$ is a differentiable function on a domain G, then every its derivative $f^{(n)}(z)$ $(n = 1, 2, \ldots)$ is differentiable on G.*

The next result serves as a converse to Cauchy's integral theorem.

Theorem 26. (Morera's Theorem) *Let $f(z)$ be a continuous function on a simply connected domain G, and suppose that*

$$\int_L f(z)dz = 0$$

for any closed rectifiable curve L contained in G. Then $f(z)$ is differentiable on G.

Proof. For $z, z_0 \in G$, the integral

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

is path-independent. Then by the argument similar to the proof of Theorem 19 (see Remark after the proof of Theorem 19), $F(z)$ is differentiable and $F'(z) = f(z)$. But then by the corollary above it follows that $F' = f$ is differentiable. □

7.5. **Cauchy Inequalities. Liouville's theorem.**

Theorem 27. (Cauchy Inequalities) *Let $f(z)$ be a differentiable function on a domain G and suppose G contains the circle $\gamma_\rho : |z - z_0| = \rho$ and its interior $I(\gamma_\rho)$. Then*

$$|f^{(n)}(z_0)| \le n! \frac{M(\rho)}{\rho^n} \quad (n = 0, 1, 2, \ldots),$$

where

$$M(\rho) = \max_{z \in \gamma_\rho} |f(z)|.$$

Proof. From Theorem 25 we immediately get that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \frac{M(\rho)}{\rho^{n+1}} 2\pi\rho = n! \frac{M(\rho)}{\rho^n}.$$

□

Note that knowledge of an upper bound $M(\rho)$ for f is sufficient to write the estimates for *all* derivatives of f. We will use these inequalities later on.

Again, note that in the case of real variable there is no, and there cannot be, such estimates. Indeed, if for $f(x) = \sin(Ax)$, $f'(x) = A\cos(Ax)$, so already the first derivative can be arbitrarily large, despite $|f|$ being bounded by 1.

One corollary of Cauchy Inequalities is the famous Liouville's theorem.

Theorem 28. (Liouville) *If $f(z)$ is differentiable on $\mathbb{C}$ and bounded, then $f(z) = const$.*

Proof. Let M be an upper bound for $|f|$ on $\mathbb{C}$. At every point $z \in \mathbb{C}$, we have by Cauchy Inequalities that $f'(z) \le \frac{M}{\rho}$ for every $\rho > 0$, so $f'(z) = 0$ and therefore f is constant. □

Remark. The essential part in the proof was that $\frac{M(\rho)}{\rho} \to 0$ as $z \to \infty$. So the proof would still work under that condition that $\frac{f(z)}{z} \to 0$ $(z \to \infty)$ instead of boundedness. It means that there are no entire functions "squeezed" between bounded and linear functions, like square root or logarithm would be.

Later we will give an even more immediate proof of Liouville's theorem and look into its consequences.

Corollary 4. (Fundamental theorem of algebra) *Every non-constant polynomial with complex coefficients $P(z)$ has at least one complex root.*

Proof. Suppose P has no complex roots. Then $|P(z)| \ne 0$. Since P is a continuous function and $P(z) \to \infty$ $(z \to \infty)$, there exists $m > 0$ such that $|P(z)| \ge m > 0$ (because $|P|$ is "large" outside some disk $|z| < R$ and has a minimum on the disk $|z| \le R$). Then the function $f(z) = \frac{1}{P(z)}$ is analytic and bounded by $\frac{1}{m}$, so by Liouville's theorem, $f(z) = const$. Then $P(z) = const$, i.e. P is of non-positive degree. □

Lecture 8. Corollaries of Cauchy Integral Formula. Functions series

March 22, 2017
Relevant Sections in Markushevich:
II.5.22, I.14.72, I.15.75

8.1. **Harmonic functions.** Now that we know that every differentiable on a domain function f is infinitely differentiable and therefore so are its real and imaginary parts, we can observe the following. Let $f(z) = u(x,y) + iv(x,y)$, where u, v are real-valued functions of real variables x, y, be analytic on a domain G. Then by Cauchy–Riemann equations and by infinite differentiability of u, v, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0,$$

that is u satisfies the Laplace equation $\Delta u = 0$ on G (where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$). Such functions are called *harmonic* on the domain G. Same computation can be done for v, so we obtain the following statement.

Theorem 29. *If $f(z) = u(x,y) + iv(x,y)$, where u, v are real-valued functions of real variables x, y, is analytic on a domain G, then u and v are harmonic on the same domain.*

Now we can answer the reasonable question, which functions $u(x,y)$ can be completed into an *analytic* function $f = u + iv$? The above theorem asserts that u must be harmonic. We show in the next theorem that this requirement is sufficient (if the domain is simply connected).

For a given harmonic u, a function v such that $u + iv$ is analytic is called a *harmonic conjugate* of u. Note that this is not exactly a symmetric relation: if v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

Example. $u(x,y) = 2xy$. Note that $\Delta u = 0$. Find its harmonic conjugate. We have $\frac{\partial v}{\partial y} = 2y$, so $v = y^2 + C(x)$. To find $C(x)$, differentiate $\frac{\partial v}{\partial x} = C'(x)$. We have $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, so $C'(x) = -2x$, and $C = x^2 + C_0$, so $v = y^2 - x^2 + c_0$. This is of course unsurprising since $f(z) = -iz^2$ is an analytic function with $\operatorname{Re} f = 2xy$.

Theorem 30. *Let $u(x,y)$ be a harmonic function on a simply connected domain G. Then there exists a unique up to an additive real constant function $v(x,y)$ harmonic on the domain G such that $f(z) = u(x,y) + iv(x,y)$ is analytic on the same domain.*

Proof. Finding such v amounts to solving a partial differential equation

$$\begin{aligned} \tfrac{\partial v}{\partial x} &= P(x,y), &\quad& \text{where } P(x,y) = -\frac{\partial u}{\partial y}, \\ \tfrac{\partial v}{\partial y} &= Q(x,y), && \text{where } Q(x,y) = \frac{\partial u}{\partial x}. \end{aligned}$$

As we know from PDEs or multivariable real analysis, in the event when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (which is the case since u is harmonic), a solution $v(x,y)$, up to an additive constant, is given by

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} P(x,y)dx + Q(x,y)dy,$$

where $(x_0, y_0), (x, y) \in G$. Note that the integral is path independent by Green's theorem and simple connectedness of G. □

Example. Consider $u(x,y) = 4x^3y - 4xy^3$. Then we have by the above formula

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (-4x^3 + 12xy^2)dx + (12x^2y - 4y^3)dy = -x^4 + 6x^2y^2 - y^4 + C_0,$$

where the computation can be done, for example, by picking a curve that consists of two straight line segments joining points (x_0, y_0), (x, y_0), (x, y); or a straight line from (x_0, y_0) to (x, y). Note that the answer is hardly surprising since $u(x, y) = \mathrm{Im}(z^4) = \mathrm{Re}(-iz^4)$.

EXAMPLE. For those who are not fans of curve integrals, the same computation can be carried out in terms of single-variable integration. Indeed, consider the same $u(x, y) = 4x^3y - 4xy^3$. Then if we are to satisfy Cauchy–Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4x^3 + 12xy^2.$$

Integrating, we get

$$v = \int (-4x^3 + 12xy^2)dx = -12x^4 + 6x^2y^2 + C(y).$$

To find $C(y)$, we check the other Cauchy–Riemann equation:

$$\frac{\partial v}{\partial y} = 0 + 12x^2y + C'(y) = \frac{\partial u}{\partial x} = 12x^2y - 4y^3,$$

So $C(y) = -y^4 + C_0$, which gives that same answer as we obtained before, $v = -x^4 + 6x^2y^2 - y^4 + C_0$. (Note that one can show that x is cancelled out in the above equation precisely because u is harmonic.)

REMARK. Note that the requirement of the domain to be simply connected is essential. If the domain is multiply connected, we may get a multivalued function v. To get an idea of what is going on in such case, look at $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$.

Observe the following. Suppose u, v are harmonic conjugates. Then the function $(u + iv)^2 = u^2 - v^2 + 2iuv$ is analytic and therefore $u^2 - v^2$ is harmonic. This can be generalized in the following theorem.

Theorem 31. *(Composition of harmonic functions) Let u, v be conjugate harmonic functions on a domain G. For any function $g = s + it$ analytic on the image of G under $u + iv$, the functions $s(u(x, y), v(x, y))$, $t(u(x, y), v(x, y))$ are harmonic on G, and the latter is a harmonic conjugate of the former.*

Proof. Denote $u + iv = f$. Then $g \circ f$ is analytic on G by chain rule. The statement follows by Theorem 29. $\square$

For example, if u, v are harmonic conjugates, then so is uv, and its harmonic conjugate is $-\frac{1}{2}(v^2 - u^2)$.

8.2. **Change of variable in the complex integral.** Another thing that we are equipped to deal with now that we know that analytic functions are infinitely differentiable is change of variable in the complex integral.

Theorem 32. (Change of variable in the complex integral) *Let $f(z)$ be a differentiable function on a domain G, let L be a rectifiable curve contained in G, with image $\Lambda = f(L)$ under the map $z \to w = f(z)$. Then Λ is a rectifiable curve and*

$$\int_\Lambda \Phi(w)dw = \int_L \Phi(f(z))f'(z)dz,$$

where $\Phi(w)$ is any continuous function on Λ.

Proof. First we need to show that both integrals exist. (Note that before we established Corollary 2, we couldn't even assert that the integral on the right hand side exists since f' could hypothetically ruin integrability. In real variable, there are plenty such functions, e.g., $f(x) = x^2 \sin(1/x^2)$ is differentiable but its derivative is unbounded and therefore not integrable on a neighborhood of 0.)

Since f is differentiable, by Corollary 2 so is f', and, in particular, f' is continuous on L. Therefore the integral $\int_L \Phi(f(z))f'(z)dz$ exists.

To show that the left hand side integral exists, we show that Λ is rectifiable. Let $\mathcal{P}$ be a partition of domain of function parameterizing L, let $z_0, \ldots, z_n$ be points on L corresponding to partition points of $\mathcal{P}$, and let $w_k = f(z_k)$, $k = 0, 1, \ldots, n$. Then

$$\sup_{\mathcal{P}} \sum_{k=1}^{n} |w_k - w_{k-1}| = \sup_{\mathcal{P}} \sum_{k=1}^{n} \left| \int_{\sigma_k} f'(z)dz \right| \leq M \sum_{k=1}^{n} l_k = Ml,$$

where σ_k is the piece of L joining z_{k-1} and z_k, l_k is the length of σ_k, l is the length of L, and

$$M = \max_{z \in L} f'(z).$$

The latter maximum is finite because f' is continuous and L is a closed bounded set. Therefore,

$$\sup_{\mathcal{P}} \sum_{k=1}^{n} |w_k - w_{k-1}| < \infty$$

and Λ is rectifiable.

Second, to prove the equality between integrals, we note that by fundamental theorem of calculus,

$$\begin{aligned} \int_\Lambda \Phi(w)dw &= \lim_{|\mathcal{P}| \to 0} \sum_{k=1}^{n} \Phi(w_k)(w_k - w_{k-1}) = \\ &= \lim_{|\mathcal{P}| \to 0} \sum_{k=1}^{n} \Phi(w_k) \int_{\sigma_k} f'(z)dz = \\ &= \lim_{|\mathcal{P}| \to 0} \sum_{k=1}^{n} \int_{\sigma_k} \Phi(f(z_k))f'(z)dz \end{aligned}$$

while

$$\int_L \Phi(f(z))f'(z)dz = \lim_{|\mathcal{P}| \to 0} \sum_{k=1}^{n} \int_{\sigma_k} \Phi(f(z))f'(z)dz.$$

Then

$$\begin{aligned} \int_\Lambda \Phi(w)dw &- \int_L \Phi(f(z))f'(z)dz = \\ &= \lim_{|\mathcal{P}| \to 0} \sum_{k=1}^{n} \int_{\sigma_k} [\Phi(f(z_k)) - \Phi(f(z))]f'(z)dz. \end{aligned}$$

Note that since $\Phi(f)$ is continuous on L, it is uniformly continuous, so for partitions fine enough, given arbitrary $\varepsilon > 0$ we have

$$|\Phi(f(z_k)) - \Phi(f(z))| < \varepsilon$$

for all $k = 1, 2, \ldots, n$. Therefore, for fine enough partitions

$$\left| \sum_{k=1}^{n} \int_{\sigma_k} [\Phi(f(z_k)) - \Phi(f(z))]f'(z)dz \right| \leq M\varepsilon l,$$

so $\int_\Lambda \Phi(w)dw - \int_L \Phi(f(z))f'(z)dz = 0$. ∎

8.3. **Function series.** Now we start next major topic, studying analytic functions through their Taylor series. At this point we know that every complex differentiable on an open set function is infinitely differentiable on the same set, by Corollary 2. By itself, that does not mean that the function is represented by its Taylor series. For example, this is not always the case for infinitely differentiable real functions, as we can see by considering the function $f(x) = e^{-1/x^2}$, extended to 0 by $f(0) = 0$. It is not hard to see that this is a infinitely differentiable function with $f^{(n)}(0) = 0$, so its Taylor series at $x = 0$ is 0.

Our goal for the moment is to show that every complex differentiable on an open set function is indeed represented by its Taylor series, which will finally justify the use of the term *analytic* function. For that, in the remaining part of this lecture we recall some results about function series.

Let

$$\sum_{n=0}^{\infty} f_n(z) = f_1(z) + f_2(z) + \ldots + f_n(z) + \ldots$$

be an infinite series of functions defined on a set $E \subseteq \mathbb{C}$. By $s_n(z)$, $n = 0, 1, 2, \ldots$, we denote n-th partial sum of this series:

$$s_n(z) = f_0(z) + f_1(z) + \ldots + f_n(z).$$

Definition 15. *The series* $\sum_{n=1}^{\infty} f_n(z)$ *is said to* converge *to a function* $f(z)$ uniformly *on* E *if given any* $\varepsilon > 0$*, there exist an integer* $N > 0$ *such that*

$$|f(z) - s_n(z)| < \varepsilon$$

for any integer $n > N$ *and any point* $z \in E$*.*

Theorem 33. (Cauchy Criterion) *Series* $\sum_{n=0}^{\infty} f_n(z)$ *converges uniformly on* E *if given any* $\varepsilon > 0$*, there exists an integer* $N > 0$ *such that*

$$|s_n(z) - s_m(z)| < \varepsilon$$

for any integers $m, n > N$ *and any point* $z \in E$*.*

Proof. See calculus. □

Remark. Of course, we may consider series where terms are indexed by numbers starting from any integer n_0, for example, $\sum_{n=3}^{\infty} f_n$.

Example. Consider power series $\sum_{n=0}^{\infty} z^n$ on sets $E_1 = \{z \mid |z| < 1/2\}$, $E_2 = \{z \mid |z| < 1\}$, $E_3 = \{z \mid |z| < 10\}$.

On E_1 the series converges uniformly, as will be evident from the next theorem. On E_3 the series diverges. On E_2 the series does converge as a geometric series with ratio z, $|z| < 1$. Show that it does not converge uniformly on E_2. Indeed,

$$|s_{n+p}(z) - s_n(z)| = |z^{n+1} + \ldots + z^{n+p}| = \frac{|z|^{n+1}|1 - z^p|}{|1 - z|} \geq \frac{|z|^{n+1}|1 - |z|^p|}{|1 - z|}.$$

Take $p = n$ and $z = \frac{n-1}{n}$.

$$|s_{2n}(z) - s_n(z)| \geq n\left(1 - \frac{1}{n}\right)^{n+1}\left(1 - \left(1 - \frac{1}{n}\right)^n\right) \to \infty$$

since $\left(1 - \frac{1}{n}\right)^n \to e^{-1}$. Therefore, Cauchy criterion fails for this series on E_2.

Theorem 34. (Weierstrass M-test) *Given a convergent number series $\sum_{n=0}^{\infty} M_n$, where $M_n \in \mathbb{R}$, $M_n \geq 0$, suppose that functions $f_0(z), f_1(z), \ldots$ are such that $|f_n(z)| \leq M_n$ for all $z \in E$ and n exceeding some fixed number $N > 0$. Then $\sum_{n=0}^{\infty} f_n(z)$ is convergent uniformly on E.*

Proof. Let $s_n(z) = f_0(z) + \ldots + f_n(z)$, and $S_n = M_0 + \ldots + M_n$. For definiteness, let $n > m$. Then

$$|s_n(z) - s_m(z)| \leq |f_{m+1}(z)| + \ldots + |f_n(z)| \leq M_{m+1} + \ldots + M_n = |S_n - S_m|.$$

The statement follows by Cauchy criterion. Work out details. □

In the above example, the uniform convergence on E_1 follows directly from this theorem with $M_n = \left(\frac{1}{2}\right)^n$.

Theorem 35. *Suppose*

$$\sum_{n=0}^{\infty} f_n(z) = f(z)$$

is uniformly convergent on E, and each $f_n(z)$ is continuous on E. Then $f(z)$ is continuous on E.

Proof. Use $|f(z) - f(z_0)| \leq |f(z) - s_n(z)| + |s_n(z) - s_n(z_0)| + |s_n(z_0) - f(z_0)|$. Work out details. □

Theorem 36. *Given a rectifiable curve L, suppose*

$$\sum_{n=0}^{\infty} f_n(z) = f(z)$$

is uniformly convergent on L, and every f_n is continuous on L. Then the series above can be integrated term by term along L, that is

$$\sum_{n=0}^{\infty} \int_L f_n(z)dz = \int_L \sum_{n=0}^{\infty} f_n(z)dz = \int_L f(z)dz.$$

Proof. First of all, f is continuous on L by the previous theorem, so the integral $\int_L f(z)dz$ is defined.

Second, since $\sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent on L, choose N such that

$$|s_n(z) - f(z)| < \varepsilon/l$$

for every $n > N$ and $z \in L$, where l is the length of L. Then for each $n > N$ we have

$$\begin{aligned} \left| \sum_{k=0}^{n} \int_L f_k(z)dz - \int_L f(z)dz \right| &= \left| \int_L \sum_{k=0}^{n} f_k(z)dz - \int_L f(z)dz \right| = \\ &= \left| \int_L (s_n(z) - f(z))\, dz \right| \leq \\ &\leq l \cdot \varepsilon/l = \varepsilon. \end{aligned}$$

□

We are now ready to prove that every analytic complex function is represented by its Taylor series. We will do that in the next lecture.

Lecture 9. Taylor Series. Weierstrass' theorem

March 29, 2017
Relevant Sections in Markushevich:
I.15.75, I.16.78–79.

9.1. **Taylor expansion.**

Theorem 37. (Taylor series) *Let function f be analytic on a domain G. Let a be a point in G, and let the circle $\gamma_\rho : |z-a| = \rho$ be contained in G together with interior $I(\gamma_\rho)$. Then $f(z)$ can be represented as a sum of series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{at every } z \in I(\gamma_\rho),$$

where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta.$$

Moreover, this series converges uniformly in $I(\gamma_\rho)$.

Proof. We have $z \in I(\gamma_\rho)$ and $\zeta \in \gamma_\rho$. Note that

$$\begin{aligned} \tfrac{1}{\zeta-z} &= \tfrac{1}{(\zeta-a)-(z-a)} = \tfrac{1}{\zeta-a} \tfrac{1}{1-\frac{z-a}{\zeta-a}} = \\ &= \tfrac{1}{\zeta-a} \sum_{n=0}^{\infty} \left(\tfrac{z-a}{\zeta-a}\right)^n = \\ &= \sum_{n=0}^{\infty} \tfrac{(z-a)^n}{(\zeta-a)^{n+1}}. \end{aligned}$$

Note that $\frac{|z-a|}{|\zeta-a|} = \frac{|z-a|}{\rho} < 1$, so by Weierstrass M-test (Theorem 34), the series above converges uniformly in $\zeta \in \gamma_\rho$. Therefore we can integrate such series term by term. We have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_\rho} \sum_{n=0}^{\infty} \frac{f(\zeta)(z-a)^n}{(\zeta-a)^{n+1}} d\zeta =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)(z-a)^n}{(\zeta-a)^{n+1}} d\zeta \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \right) (z-a)^n =$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n.$$

To prove uniform convergence of the above series, note that by Cauchy's inequalities (Theorem 27), $|c_n| = \left|\frac{f^{(n)}(a)}{n!}\right| \le \frac{M(\rho_1)}{\rho_1^n}$ for any ρ_1 such that the circle γ_{ρ_1} is contained in G together with its interior.

Then taking such $\rho_1 > \rho$, we get that $|\frac{f^{(n)}(a)}{n!}(z-a)^n| \le M(\rho_1) \left(\frac{\rho}{\rho_1}\right)^n$, so the series converges uniformly on γ_ρ by, again, Weierstrass M-test. □

The theorem above asserts, among other things, that if a function is *complex differentiable once*, then *it is infinitely differentiable* and, moreover, *equals to the sum of its Taylor series.* This justifies the use of the term *analytic* function whenever we are talking about a function, complex differentiable on an open set. (Recall that we also can use another term, *holomorphic* function.)

Corollary 5. *In the above notation,* $|c_n| \leq \frac{M(\rho)}{\rho^n}$.

Proof. Follows directly from Cauchy inequalities (Theorem 27), or from the proof above. □

Now we have another way to prove Liouville's Theorem 28. Recall that functions analytic on $\mathbb{C}$ are called *entire*.

Theorem 28 (Liouville) *If* $f(z)$ *entire and bounded, then* $f(z) = const$.

Proof. For $n \geq 1$, $|c_n| \leq \frac{M(\rho)}{\rho^n}$ for any ρ, so $|c_n| = 0$, and, therefore $f(z) = c_0$. □

So, Liouville's theorem says that a bounded entire function must be constant. Another way to state the Liouville's theorem is the following: if a function is analytic on $\overline{\mathbb{C}}$, then it is constant. (We say that a function $f(z)$ is analytic at $z = \infty$ if $f(1/z)$ is analytic at $z = 0$. At this point of the course it is not entirely clear why being bounded and being analytic at ∞ are equivalent conditions in this case, but we will see that they indeed are once we get to classification of singular points.)

9.2. **Radius of convergence of a power series.** Now that we know that every analytic function is represented by its Taylor series, there is a natural question whether the converse is true; that is, whether the sum of a power series is an analytic function. Another reason for asking this question is that, as it will be apparent in the next lecture, it is precisely what is missing to fully take advantage of Taylor expansion. To investigate this question, we first recall some classic facts about power series.

Reminder. Let (x_n) be a sequence in $\mathbb{R}$. A number $x \in \mathbb{R}$ or $+\infty$ is called an *accumulation point* (sometimes called limit point or cluster point) of (x_n) if every open neighborhood of x contains infinitely many terms of (x_n) (by a neighborhood of $+\infty$ we mean any open ray $(a, +\infty)$.)

For example, $((-1)^n)$ has two accumulation points: ± 1. A sequence that has a limit l has only one accumulation point, l. If a sequence (q_n) enumerates all rational numbers, then the set of its accumulation point is $\mathbb{R} \cup \{\pm\infty\}$.

For a sequence (x_n) in $\mathbb{R}$, its upper limit (or limit superior) $\limsup_{n\to\infty} x_n$ is the supremum of the set of accumulation (limit) points of x_n. As a convention, for our purposes, we allow $\limsup$ to be $+\infty$ (which corresponds to the sequence (x_n) being unbounded above); or $-\infty$ (which corresponds to the sequence having limit $-\infty$). One can show that the set S of limit points of a sequence is closed, so we can replace *supremum* with *maximum*.

Examples. $\limsup_{n\to\infty}(-1)^n = 1$. $\limsup_{n\to\infty}(-1)^n + \frac{1}{n} = 1$.
$\limsup(0, 1, 0, 2, 0, 3, \ldots) = \infty$. $\limsup(0, -1, -2, \ldots) = -\infty$.

Note that for a limit superior of a real sequence always exists as a real number, $+\infty$, or $-\infty$.

Theorem 38. (Cauchy–Hadamard) *Given the power series*

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots,$$

let $R = \frac{1}{\Lambda}$, *where*

$$\Lambda = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

and let γ *be the circle* $|z - z_0| = R$ *with interior* $I(\gamma)$ *and exterior* $E(\gamma)$. *Then there are three possibilities:*

(1) *If* $R = 0$*, the series diverges* $\forall z \neq z_0$.
(2) *If* $0 < R < \infty$*, the series is absolutely convergent* $\forall z \in I(\gamma)$ *and divergent* $\forall z \in E(\gamma)$.
(3) *If* $R = \infty$*, the series is absolutely convergent for all* $z \in \mathbb{C}$.

Proof. Follows from the following lemma (known as root test).

Lemma 5. *Let* $\sum\limits_{n=0}^{\infty} b_n$ *be a complex series. Let* $\ell = \limsup\limits_{n\to\infty} \sqrt[n]{|b_n|}$*. The series converges if* $\ell < 1$ *and diverges if* $\ell > 1$.

Proof of Lemma. If $\ell < 1$, take any $\ell < s < 1$. The series is ultimately dominated by $\sum s^n$, which converges.

If $\ell > 1$, take any $1 < s < \ell$. The series has infinitely many terms $> s^n$, thus fails nth term test. See calculus for details. $\square$

To prove the theorem, consider

$$\ell = \limsup_{n\to\infty} \sqrt[n]{|a_n(z - z_0)^n|} = |z - z_0|\Lambda$$

and compare this value to 1. $\square$

Circle γ is called the *circle of convergence*, $I(\gamma)$ is called the *disc of convergence*, and R is called the *radius of convergence*.

Remark. The convergence is uniform on every closed disc (therefore, every closed set) contained in $I(\gamma)$ (this will be stated in the next lecture as a separate theorem).

Remark. On the other hand, uniform convergence on the whole $I(\gamma)$ fails. We will prove later that it fails always (for nonconstant functions), but for now we at least have an explicit example (see example after Theorem 33).

Remark. Finally, note that the theorem above does not assert anything about convergence at points of the circle γ itself, and for a good reason: the set of points of γ where the series converges depends on the series and may be very complicated.

It is easy to use Taylor expansion Theorem 37 to deduce that following notable functions expand as Taylor series that converge everywhere, i.e. have infinite radius of convergence. At $z_0 = 0$ the series take the following form:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \ \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \ \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \ \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

For example, perform the necessary computations in the case of $\sin z$. Compute derivatives directly:

$$\sin^{(4k)} z = \sin z, \quad \sin^{(4k+1)} z = \cos z, \quad \sin^{(4k+2)} z = -\sin z, \quad \sin^{(4k+3)} z = -\cos z.$$

Plugging in $z_0 = 0$, we get that $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$. Now find the radius of convergence using Cauchy–Hadamard theorem (note that the computation below is not really necessary, because $\sin z$ is entire, so any ρ can be used in Taylor expansion Theorem 38). Nevertheless, we have

$$\Lambda = \limsup_{n\to\infty} \sqrt[n]{|c_n|},$$

where $c_n = 0$ for even n and $c_{2k+1} = 1/(2k+1)!$ for odd $n = 2k+1$. $\lim \sqrt[n]{0} = 0$ and, since by calculus $\lim \sqrt[n]{n^n/n!} = e$, $\lim \sqrt[2k+1]{1/(2k+1)!} = 0$, so

$$\Lambda = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \max\{0,0\} = 0,$$

and therefore $R = \infty$.

It is also easy to establish that $\frac{1}{1-z}$ expands at $z_0 = 0$ as $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, with radius of convergence 1, either by taking $\limsup \sqrt[n]{1} = 1$, or by arguing that the largest open disc centered at 0 that fits in $\mathbb{C} \setminus \{1\}$ is of radius 1. (Notice that neither argument is really necessary for this series, as it is just the sum of a geometric series.)

We will deal with expansion of arbitrary rational function $P(z)/Q(z)$ later. For now, we can give an instructive example. If $f(z) = \frac{az+b}{(z-c)(z-d)}$ with $c, d \neq 0$ and $c \neq d$, then we can decompose

$$f(z) = \frac{az+b}{(z-c)(z-d)} = \frac{A}{z-c} + \frac{B}{z-d} = \frac{-A}{c} \cdot \frac{1}{1-\frac{z}{c}} + \frac{-B}{d} \cdot \frac{1}{1-\frac{z}{d}}.$$

The two latter functions can be expanded using the Taylor series for $1/(1-z)$. Radius of convergence is in this case $\min\{|c|,|d|\}$, as it follows from problems of Homework 9, or as can be seen by a direct computation according to Cauchy–Hadamard Theorem 38, or by application of Taylor expansion Theorem 37.

As one more example of applying Cauchy–Hadamard theorem, consider the series $\sum n^n z^{n^n}$. The sequence $\sqrt[n]{|c_n|}$ has terms 0 and $\sqrt[n^n]{n^n}$ (*not* $\sqrt[n]{n^n} = n$). Notice that $\lim \sqrt[n^n]{n^n} = 1$, so $\Lambda = \max\{0,1\} = 1$, and $R = 1$.

Finally, we include a brief note about Ratio Test. One may recall that, for a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, whenever the limit $\Lambda = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, it gives the radius of convergence $R = 1/\Lambda$. The proof is based on the same idea as that of "root test" Theorem 38, that is comparing to a geometric series. Applying the ratio test is often easier than root test, but it is not hard to observe that ratio test may fail irreparably even in a simple situation (see example below), while Cauchy–Hadamard Theorem 38 based on root test works *always*.

Example. Consider the series $\sum_{n=0}^{\infty} (3+(-1)^n)^n z^n$. Applying Cauchy–Hadamard Theorem we get

$$\Lambda = \limsup_{n\to\infty} \sqrt[n]{(3+(-1)^n)^n} = \limsup\,(2,4,2,4,\ldots) = 4,$$

so $R = \frac{1}{4}$. Now construct sequence $\left(|\frac{a_{n+1}}{a_n}|\right)$ and try to apply ratio test,

$$\left(\left|\frac{a_{n+1}}{a_n}\right|\right) = \left(\frac{4^2}{2}, \frac{2^3}{4^2}, \frac{4^4}{2^3}, \ldots, \frac{4^{2k}}{2^{2k-1}}, \frac{2^{2k+1}}{4^{2k}}, \ldots\right).$$

We see that the subsequence of odd terms converges to 0 and that of even terms to ∞. Therefore not only the limit does not exist, but also $\Lambda = 4$ is not even among the accumulation points of the sequence. Further, *for any* given $0 \le \Lambda \le \infty$ it is not hard to similarly construct a series for which ratio test produces a sequence with accumulation points $0, \infty$. So accumulation points of the sequence in the ratio test may not be enough to extract *any* information about the radius of convergence. In this sense, ratio test fails irreparably for such series, as stated above.

9.3. **Weierstrass' theorem on uniformly convergent series of analytic functions.** Recall that a subset $K \subseteq \mathbb{C}$ is called *compact* if it is closed and bounded.

Theorem 39. (Weierstrass' theorem on uniformly convergent series of analytic functions) *If the series*

$$\sum_{n=0}^{\infty} f_n(z) = f(z)$$

is uniformly convergent on every compact subset of a domain G, and if every term $f_n(z)$ is analytic on G, then $f(z)$ is analytic on G. Moreover, the series can be differentiated term by term any number of times, i.e.

$$\sum_{n=0}^{\infty} f_n^{(k)}(z) = f^{(k)}(z) \quad \forall z \in G, \forall k \geq 0$$

and each differentiated series is uniformly convergent on every compact subset of G.

Proof. Let $z_0 \in G$. Pick $\gamma : |z - z_0| = \rho$ such that $I(\gamma) \cup \gamma \subseteq G$. Multiplying the equality

$$\sum_{n=0}^{\infty} f_n(z) = f(z)$$

by $\frac{k!}{2\pi i (z-z_0)^{k+1}}$, we get

$$\frac{k!}{2\pi i} \sum_{n=0}^{\infty} \frac{f_n(z)}{(z-z_0)^{k+1}} = \frac{k!}{2\pi i} \frac{f(z)}{(z-z_0)^{k+1}}.$$

Show that the convergence of the above series is uniform on γ. Indeed, it follows by Cauchy criterion (Theorem 33):

$$\left| \sum_{n=N}^{M} f_n(z)/(z-z_0)^k \right| = \frac{1}{|z-z_0|^k} \left| \sum_{n=N}^{M} f_n(z) \right| = \frac{1}{\rho^k} \left| \sum_{n=N}^{M} f_n(z) \right| \to 0 \ (M, N \to \infty).$$

By Theorem 36 this series can be integrated term-wise:

$$\frac{k!}{2\pi i} \sum_{n=0}^{\infty} \int_\gamma \frac{f_n(z)}{(z-z_0)^{k+1}} dz = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz.$$

Since all f_n are analytic, by Cauchy Integral Formula (Theorem 22), $\frac{k!}{2\pi i} \int_\gamma \frac{f_n(z)}{(z-z_0)^{k+1}} dz = f_n^{(k)}(z_0)$, so

$$\sum_{n=0}^{\infty} f_n^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz.$$

For $k = 0$, we have

$$\sum_{n=0}^{\infty} f_n(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0} dz.$$

Note that the left hand side is equal to $f(z_0)$, so

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0} dz,$$

where $f(z)$ (in the integrand) is continuous on γ as a uniform (on γ) limit of continuous functions. We get that $f = \sum_{n=0}^{\infty} f_n$ is represented by an integral of Cauchy type. By Theorem 25 f is, therefore, differentiable.

Since we now know that f is differentiable, for $k > 0$ we get that righthand side is equal to $f^{(k)}(z_0)$, so we have

$$\sum_{n=0}^{\infty} f_n^{(k)}(z) = \frac{k!}{2\pi i} \sum_{n=0}^{\infty} \int_\gamma \frac{f_n(z)}{(z-z_0)^{k+1}} dz = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz = f^{(k)}(z).$$

To prove the last part of the statement, the uniformity, we need the following lemma.

Lemma 6. *Convergence of a series $\sum g_n(z)$ is uniform on compact subsets of a domain G if and only if every point of G has a neighborhood on which the convergence is uniform.*

Proof. The $\Rightarrow$ direction follows immediately since by considering a closed (therefore compact) neighborhood of a given point.

The other direction $\Leftarrow$ follows by Heine–Borel theorem: given a compact set K, each its point possesses a neighborhood on which the convergence is uniform. By compactness (Heine–Borel theorem), K can be covered by finitely many such neighborhoods, which is enough for uniformity on the whole K. $\square$

By the lemma above, to establish that the differentiated series is uniformly convergent on every compact subset of G, it is enough to show that every point $z_0 \in G$ has a neighborhood on which the series converges uniformly. For a point z_0, consider the open disc $B_{\rho/2}(z_0)$. Pick n such that $|s_n(z) - f(z)| < \varepsilon$ on γ (here s_n stands for the n-th partial sum $\sum_{j=0}^{n} f_j(z)$). Then for any $z \in B_{\rho/2}(z_0)$, we have

$$\left| \sum_{j=0}^{n} f_j^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \sum_{j=0}^{n} \int_\gamma \frac{f_j(\zeta)}{(\zeta - z)^{k+1}} d\zeta - \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right| =$$

$$= \left| \frac{k!}{2\pi i} \int_\gamma \frac{s_n(z) - f(z)}{(\zeta - z)^{k+1}} d\zeta \right| \le \frac{k!}{2\pi} \frac{\varepsilon}{(\rho/2)^{k+1}} 2\pi\rho \to 0$$

as $\varepsilon \to 0$. (The denominator part of the last inequality is provided by $|\zeta - z| \ge \rho/2$ since $\zeta \in \gamma$ and $z \in B_{\rho/2}(z_0)$.) $\square$

Note that G being a domain is essential, as the following two examples show. (They also demonstrate that the statement of Weierstrass theorem fails in $\mathbb{R}$.)

Example 1. $\varphi(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$, where $x \in \mathbb{R}$, $0 < b < 1$, $ab > 1 + \frac{3\pi}{2}$ is uniformly convergent (dominated by $\sum b^n$), but one can show that the function $\varphi(x)$ is not differentiable at any point of $\mathbb{R}$. This example shows that the first part of the statement (analyticity) can fail if G is not a domain. (Note that the limit is still continuous as a uniform limit of continuous functions. So this is also an example of a continuous but nowhere differentiable function.)

Example 2. Series

$$\sin x + \left(\frac{\sin 2x}{2} - \sin x \right) + \left(\frac{\sin 3x}{3} - \frac{\sin 2x}{2} \right) + \dots$$

is uniformly convergent to 0 on $\mathbb{R}$ (because $s_n \le 1/n$), but differentiating the series termwise gives a divergent series $\cos x + (\cos 2x - \cos x) + (\cos 3x - \cos 2x) + \dots$. This example shows that the second part of the statement (term-wise differentiation) can fail if G is not a domain.

Switching from series to sequences, we get the following corollary.

Corollary 6. (Weierstrass' theorem on uniformly convergent sequences of analytic functions) *If the sequence $(g_n(z))$ uniformly converges to $g(z)$ on every compact subset of a domain G, and if every term $g_n(z)$ is analytic on G, then $g(z)$ is analytic on G. Moreover, the sequence can be differentiated term by term any number of times, i.e. $(g_n^{(k)}(z)) \to g^{(k)}(z)$, and each differentiated sequence converges uniformly on every compact subset of G.*

Proof. Formally put $g_0 = 0$ and define $f_n(z) = g_n(z) - g_{n-1}(z)$. The statement now follows by Weierstrass' Theorem 39. □

Lecture 10. Uniqueness theorems. Maximum modulus principle

April 5, 2017
Relevant Sections in Markushevich:
I.16.79, I.17.82-83

10.1. **Sum of a power series is analytic.** Now, note that if $\gamma : |z - z_0| = R$ is the circle of convergence of a power series $a_0 + a_1(z - z_0) + \ldots$, then the series is uniformly convergent on every compact subset of $I(\gamma)$ by Weierstrass M-test, so Weierstrass theorem 39 applies to convergence of a power series on the disc $I(\gamma) = \{|z - z_0| < R\}$.

Remark. Uniform convergence on $I(\gamma)$ fails. We will prove later that it fails always (for nonconstant functions), but for now we at least have an explicit example (see example after Theorem 33).

Recall that so far we know that an analytic function can be represented by its Taylor series. What about other way around? That is, given a power series, can we assert that its sum is an analytic function? The answer is yes. More exactly, the following theorem holds.

Theorem 40. *The power series*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

with radius of convergence R defines an analytic function $f(z)$ on the disc K : $|z - z_0| < R$, and the coefficients $a_0, a_1, \ldots$ are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Moreover, the series $f(z)$ can be differentiated term by term any number of times, and each differentiated series converges uniformly on compact subsets of K.

Proof. Since the convergence is uniform on compact subsets of K (as mentioned above), the assertion follows by Weierstrass' Theorem 39. □

One important consequence is the following statement.

Theorem 41. *If f is analytic on a domain G, and $f(z_0) = 0$ for a point $z_0 \in G$, then $g(z) = \frac{f(z)}{z - z_0}$ (extended to $z = z_0$ by continuity) is analytic on the domain G.*

Proof. Analyticity of g at all points of G except for $z = z_0$ follows immediately from rules of differentiation. Analyticity of g at $z = z_0$ follows from the theorem above and the fact that the radii of convergence of

$$\sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ and } \sum_{n=1}^{\infty} a_n(z-z_0)^{n-1}$$

coincide. $\square$

Note that while this theorem is easy to take for granted, it is not obvious (at least, not without the machinery we developed). For example, if we consider a very "tame" function $f(z) = \overline{z}$, we can see that $f(0) = 0$, but $f(z)/z$ cannot be even extended continuously to $z = 0$ (for example, because $f(x)/x = 1$ for $0 \neq x \in \mathbb{R}$ and $f(iy)/(iy) = -1$ for $0 \neq y \in \mathbb{R}$).

Also note that for a differentiable function $\mathbb{R} \to \mathbb{R}$ s.t. $f(0) = 0$, the ratio $f(x)/x$, even if defined by continuity at 0, can easily be non-differentiable at 0, e.g. $f(x) = x|x|$.

10.2. **Uniqueness theorems.**

Theorem 42. *If the sums of two power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \ \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

coincide in a neighborhood of z_0, then $a_n = b_n$ for all $n \geq 0$.

Proof. It's a direct consequence of Theorem 40, since all derivatives at z_0 are uniquely determined by values of a function on an arbitrarily small neighborhood of z_0. $\square$

Theorem 43. *If the sums of two power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \ \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

coincide on a sequence (z_n) of distinct points such that $z_n \to z_0$, then $a_n = b_n$.

Proof. Since the sum of a power series is a continuous function within the radius of convergence, we have that

$$a_0 = \lim_{n\to\infty}(a_0 + a_1(z_n - z_0) + \ldots) = \lim_{n\to\infty}(b_0 + b_1(z_n - z_0) + \ldots) = b_0.$$

Proceed by subtracting $a_0 = b_0$ from both series and dividing by $(z - z_0)$. $\square$

Example 1. Does there exist a nonzero entire function f such that $f(1/n) = 0$ for each positive integer n? No. Indeed, set $z_n = 1/n$ and $z_0 = 0$ in the above theorem. We get that all Taylor coefficients of $f(z)$ and of 0 are the same, so $f = 0$.

Example 2. Does there exist a nonzero analytic on $\{\operatorname{Re} z > 0\}$ function f such that $f(1/n) = 0$ for each positive integer n? Yes, for example $f(z) = \sin(\frac{\pi}{z})$. This does not contradict the above theorem, since f does not have a Taylor expansion at 0 (the limit of $1/n$). (In other words, since the limit of $1/n$ *is not in the domain* of analyticity of f.)

To generalize the above two examples in the next theorem, we recall a topological notion. (The following definition works for an arbitrary topological space, but we are going to need it only for $\mathbb{C}$.)

Given a set $E \subseteq \mathbb{C}$, a point $z \in \mathbb{C}$ is called a *limit* (or *accumulation*, or *cluster*) point of E if any open neighborhood of z contains a point of E, distinct from z itself. This is equivalent to z being a limit of a sequence of distinct points in E.

It is not hard to see that the set of all limit points of a given set E is always closed.

Theorem 44. (Interior uniqueness theorem) *Suppose two functions analytic on a domain G coincide on a set that has a limit point in G. Then f and g coincide on the whole domain G.*

Proof. Consider the set $E = \{z \in G \mid f(z) = g(z)\}$. Let E_1 be the set of all limit points of E in G. Note that E_1 is closed in G since it is a set of limit points. Note also that it is open by the previous theorem. Indeed, let z_0 be a limit point of E, so there is a sequence $z_n \to z_0$ with $z_n \in E$. By the previous theorem, Taylor expansions of f and g at z_0 are equal, so E, and therefore E_1, contains an open disc centered at z_0, as required.

Now, recall that by definition of a domain, G is a connected set, so any subset of G that is both open and closed in G must be either empty or the entire G. We are given that E_1 contains at least one point, so the former is impossible, and therefore $E_1 = G$. It is only left to note that since both f and g are continuous, $f = g$ on E_1, so $f = g$ on G. □

This theorem is quite strong: it allows to assert that two analytic functions are equal on a domain G provided *only* that they are equal on a some set E that possesses a limit point in G.

Example 1. Is there a complex differentiable on $\mathbb{C}$ function $f(z)$ s.t. $f(x) = 0$ for $x \le 0$ and $f(x) = e^{-1/x^2}$ for $x > 0$? No, because $f(z)$ would have to coincide with 0.

Example 2. Is there a complex differentiable on $\mathbb{C}$ function $f(z)$ s.t. $f(x) = \sin x$ for $x > 0$ and $f(x) = \cos x - 1 + x$ if $x \le 0$? No, because $f(z)$ would have to coincide with both $\sin z$ and $\cos z - 1 + z$.

Example 3. We can now prove the addition formulas (6),(7) for cos and sin without any computations, granted that we know them for the real variable. For example, prove the formula (6): $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

Indeed, if z_1 is fixed and *real*, the left hand side and right hand side are functions of a complex variable z_2, and we know that these functions coincide if z_2 is real. By the Interior uniqueness theorem 44, we conclude that these functions must be the same for *all complex* z_2. Now, fix arbitrary complex z_2 and consider both sides as functions of a complex variable z_1. We already know that if z_1 is real, these functions coincide. Therefore, by Interior uniqueness theorem they coincide for *all complex* z_1. Recall that z_2 was arbitrary complex, so the left hand side and the right hand side coincide for all complex z_1, z_2.

Example 4. The uniqueness part of Theorem 10 (existence and uniqueness of the exponential) now follows immediately. So the proof condenses to "consider $f(z) = e^x(\cos y + i \sin y)$. It satisfies all conditions of the theorem, and it is unique by the Interior uniqueness theorem".

10.3. **Maximum Modulus Principle.** Recall that as an immediate consequence of Cauchy integral formula, we proved Theorem 24 which asserted that a differentiable on a domain function cannot have a local strict maximum of its absolute value at any point of the domain. Now, as promised, we can prove the same about non-strict maximum.

Theorem 45. (Maximum Modulus Principle) *If $f(z)$ is a nonconstant analytic function on a domain G, then $|f(z)|$ cannot have a non-strict local maximum at any point of G.*

Proof. Suppose not, that is at some point z_0 we have a neighborhood $B_r(z_0)$ such that for any $z \in B_r(z_0)$,

$$|f(z_0)| \geq |f(z)|.$$

Expand f into a Taylor series at z_0:

$$f(z) = a_0 + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots$$

where $a_0 = f(z_0) \neq 0$ (otherwise $|f(z_0)| = 0$ so $f = 0$ on $B_r(z_0)$ and, by uniqueness theorem, $f = 0$ on G), and a_k is the first nonzero coefficient among $a_1, a_2, \dots$ (if there are none, $f(z) = const$). Write

$$f(z) = f(z_0) + B(z - z_0)^k(1 + \varphi(z)),$$

where $B = a_k$ and $\varphi(z) = \frac{1}{B}(a_{k+1}(z - z_0) + a_{k+2}(z - z_0)^2 + \dots)$, i.e. φ is continuous (in fact, analytic) on G and $\varphi(z_0) = 0$. Pick $z \in B_r(z_0), z \neq z_0$ so that

$$\operatorname{Arg} B(z - z_0)^k = \operatorname{Arg} f(z_0)$$

and r so that $\varphi(z)$ is small enough ($|\varphi(z)| < 1/10$ suffices). Then

$$|f(z)| = |f(z_0) + B(z - z_k)^k(1 + \varphi(z))| > |f(z_0)|,$$

(see Figure 3) contradicting the initial assumption. $\square$

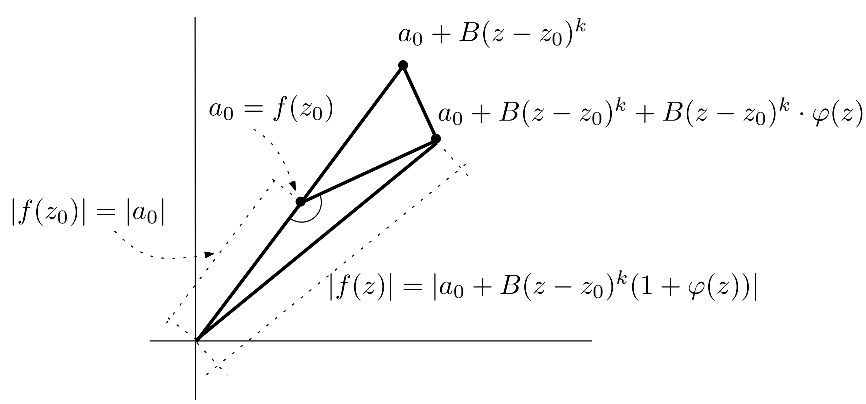

FIGURE 3. To guarantee $|f(z)| > |f(z_0)|$, we only need the marked angle to be obtuse, for which $\varphi(z) < 1/10$ is certainly enough.

Another way to put the above theorem is to say that if an analytic function has a non-strict maximum of its absolute value at a point of G, then the function is constant. Yet another way to say the same is the following: an analytic function cannot reach local non-strict maximum of its absolute value at an *interior* point of a set.

A similar proof works for so-called minimum modulus principle (with the difference that one has to make sure the argument of $B(z - z_0)^k$ is opposite of the argument of a_0). Alternatively, it can be deduced as a direct corollary of the maximum modulus principle.

Corollary 7. (Minimum Modulus Principle) *The absolute value of a nonconstant analytic function on a domain G cannot have a minimum at any point of G, which is not a zero of $f(z)$.*

Proof. Apply Maximum Modulus Principle to $1/f(z)$. □

10.4. **Schwarz's Lemma.**

Theorem 46. (Schwarz's Lemma) *Let $f(z)$ be a function which is analytic on the disc $K : |z| < R$, $f(0) = 0$ and suppose that*

$$|f(z)| \leq M < \infty$$

for all $z \in K$. Then

$$|f(z)| \leq \frac{M}{R}|z|$$

for all $z \in K$, and

$$|f'(0)| \leq \frac{M}{R}.$$

Either equality is achieved if and only if f is a linear function

$$f(z) = e^{i\alpha}\frac{M}{R}z,$$

where $\alpha \in \mathbb{R}$.

Proof. Note that function $\varphi(z) = f(z)/z$ is analytic on K. Consider a disc K_ρ : $|z| < \rho$ with a $\rho < R$. Denote by γ_ρ the circle $|z| = \rho$. Then by Maximum Modulus Principle (Theorem 45), $|\varphi|$ cannot achieve it's maximum on $\overline{K_\rho}$ at a point of K_ρ, so

$$\max_{z\in\overline{K_\rho}} |\varphi(z)| = \max_{z\in\gamma_\rho} |\varphi(z)| = \max_{z\in\gamma_\rho} \left|\frac{f(z)}{\rho}\right| = \frac{M(\rho)}{\rho} \leq \frac{M}{\rho}.$$

Note that this inequality holds for every $\rho < R$, so it also holds for $\rho = R$. But this is exactly what the theorem asserts:

$$\frac{|f(z)|}{|z|} \leq \frac{M}{R}.$$

Further, $f'(0) = a_1 = \varphi(0)$, so in particular,

$$f'(0) \leq \frac{M}{R}.$$

By the equality part of Maximum Modulus Principle (Theorem 45), equalities are only achieved for a constant $\varphi(z)$, that is, for $f(z) = C \cdot z$. In that case, $|C|$ must be equal to M/R, so $f(z) = e^{i\alpha}\frac{M}{R}z$. □

This fact has following geometric interpretation. If image of a disc K under an analytic map is inside a disc L, then the image of a smaller disc K' under the same map is a proportionally smaller disc L' inside L.

Recall that every Möbius transformation of the form

$$w(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$$

with $\alpha \in \mathbb{R}$ and $|a| < 1$ sends the closed unit disc $|z| \leq 1$ to itself (this can be checked by verifying that $|w(z)| = 1$ whenever $|z| = 1$). Schwarz's Lemma has a really nice consequence asserting that there are *no other bijective analytic* maps from unit disc to itself.

Theorem 47. *Let K be the closed unit disc $|z| \leq 1$. Suppose map $f : K \to K$ is analytic on the open disc $|z| < 1$, continuous on K, and bijective on the open disc. Then f is of the form*

$$f(z) = e^{i\alpha} \frac{z-a}{1-\bar{a}z}$$

for some $\alpha \in \mathbb{R}$ and $a \in \mathbb{C}$, $|a| < 1$.

Proof. Suppose $f(0) = a$. Consider Möbius map

$$\varphi(z) = \frac{z-a}{1-\bar{a}z}.$$

Note that composition $F(z) = \varphi(f(z))$ sends the open disc $\{|z| < 1\}$ bijectively onto itself, and, moreover, $F(0) = 0$. That is, F satisfies conditions of Schwarz's Lemma with $M = 1$ and $R = 1$. Therefore,

$$|F(z)| \leq |z|$$

for all $z \in K$ (the inequality for $|z| = 1$ follows by continuity). Note that the map F^{-1} is well defined and also satisfies conditions of Schwarz's Lemma, therefore,

$$|F^{-1}(w)| \leq |w|$$

for all $w \in K$. Comparing these two inequalities, we conclude that $|F(z)| = |z|$. By Schwarz's Lemma, this is only possible if $F(z) = e^{i\alpha} z$. So,

$$f(z) = \varphi^{-1}(e^{i\alpha} z) = e^{i\alpha} \frac{z-b}{1-\bar{b}z}$$

with $b = -ae^{-i\alpha}$. □

Lecture 11. Singular points. Laurent series

April 12, 2017
Relevant Sections in Markushevich:
I.17.84, II.1.1

11.1. **Singular points and the radius of convergence of Taylor series.** Note that if a function f is analytic in the disc $|z - z_0| < R$, then by Theorem 37 the radius of convergence of Taylor series of f at z_0 is at least R. On the other hand, if we for any reason know that function f can not be defined as an analytic function on a disc $|z - z_0| < R'$, then the radius of convergence of Taylor series of f at z_0 is less than R'. (The same argument applies to determine r and R for Laurent series.)

For example, consider $f(z) = 1/(1 + z^2)$. It is not hard to find directly that $R = 1$, but it can be seen immediately since f takes infinite value at $\pm i$. By the way, this gives a coherent reason why Talyor series of a real-valued function $1/(1 + x^2)$ diverges at $x = \pm 1$, while the function itself is perfectly fine at $x = \pm 1$ and the rest of the real line.

Another example is $f(z) = \int_0^\infty e^{-zt} dt$. The integral converges in $\operatorname{Re} z > 0$ and diverges in $\operatorname{Re} z < 0$. A direct computation shows that $f(z) = \frac{1}{z}$ for $z \in \{\operatorname{Re} z > 0\}$. So, at a point $z_0 \in \{\operatorname{Re} z > 0\}$, the radius of convergence of Taylor series is not $\operatorname{Re} z_0$, as would appear from the fact that the integral diverges for $\operatorname{Re} z < 0$, but rather $|z_0|$ since $f(z)$ can be extended analytically by $f(z) = \frac{1}{z}$ to $\mathbb{C} \setminus 0$.

The latter example highlights the following issue: given a function f on a disc $|z - z_0| < R$, the condition whether the function can be extended analytically to a larger disc $|z - z_0| < R + \varepsilon$ may be hard to check. Below we make an effort to replace it with a local condition, so that we do not have to worry about the whole annular strip $R < |z - z_0| < R + \varepsilon$ at once.

Let K be a disc $\{|z - z_0| < R\}$, and Γ its circle $\{|z - z_0| = R\}$. Given a function f on the disc K and a point $a \in \Gamma$, we say that the point a is a *regular* point w.r.t. the pair (f, K) if there is a function f_a on the disc $K_a = \{|z - a| < \rho\}$ for some $\rho > 0$ s.t. f_a is analytic and $f_a = f$ on the intersection $K \cap K_a$ (in other words, if f extends analytically to a neighborhood of a). Otherwise, we say that $a \in \Gamma$ is *singular* w.r.t. the pair (f, K).

NOTE. If a is a singular point w.r.t. the pair (f, K), then it is a singular point w.r.t. any other pair (f, K') if a happens to be on the boundary circle of K'. We therefore can call such points just *singular* points of f, without specifying a disc K.

Turns out, a function can be extended analytically beyond a circle K if and only if every point is regular. More specifically, the following theorem holds.

Theorem 48. *A function given by a power series at z_0 with radius of convergence R has at least one singular point on the circle $|z - z_0| = R$.*

Proof. Let f be given by a Taylor series on its disc of convergence $K = \{|z - z_0| < R\}$, and suppose every point on the circle $\Gamma = \{|z - z_0| = R\}$ is regular w.r.t. (f, K).

Take a union of all circles K_a with K: put $\tilde{K} = K \cup \bigcup_{a \in \Gamma} K_a$. Define a function $\tilde{f} : \tilde{K} \to \mathbb{C}$ by setting $\tilde{f}(z) = f(z)$ if $z \in K$, and $\tilde{f}(z) = f_a(z)$ if $z \in K_a$. We have to show consistency of this definition. Suppose some z is in both K_a and K_b. Then both f_a and f_b are equal to f on the triple intersection $K_a \cap K_b \cap K$, so by interior uniqueness theorem 44 $f_a = f_b$ on $K_a \cap K_b$, in particular $f_a(z) = f_b(z)$.

Therefore, we extended f analytically to $\tilde{K}$. Now we have to show that $\tilde{K}$ actually contains a disc larger than K. This can be done by switching to finite union by compactness of Γ, or by employing the notion of distance between sets. Indeed, note that $\tilde{K}$ is open, so $\mathbb{C} \setminus \tilde{K}$ is a closed set that does not intersect the closed set $K \cup \Gamma$, therefore (see Lemma 3) $\varepsilon = \mathrm{dist}(\mathbb{C} \setminus \tilde{K}, K \cup \Gamma) > 0$. Therefore, the disc $K' = \{|z - z_0| < R + \varepsilon\}$ is contained in $\tilde{K}$, and f extends analytically to K', so K is not the disc of convergence. $\square$

This theorem gives a practical way to find a radius of convergence of Taylor series of elementary functions (and other functions given in a reasonable way): one only needs to find distance from the center z_0 to the closest singular point.

EXAMPLE. $f(z) = 1/\sin z$, $z_0 = 6 + 7i$. Finding radius of convergence through explicit application of Cauchy–Hadamard theorem is unpleasant here, because the Taylor series itself is unpleasant to find (not impossible though). Instead, we note that f is analytic everywhere except for zeros of sin, i.e. the points $z_k = 2\pi k$, and takes infinite value at those points. So the radius of convergence of Taylor series is the minimum of distances $|z_0 - z_k|$. In this case, it's $\sqrt{(6 - 2\pi)^2 + 7^2}$.

11.2. **Laurent series.** Now we start looking into behavior of functions at points of non-analyticity. One of important tools for that are Laurent series, that is, power series where negative powers of the variable are allowed. General idea is that we already developed all the machinery required to deal with such series when we were studying Taylor series. All we have to do is carefully apply those ideas and techniques.

Theorem 49. *Given a function series*

$$A_0 + A_1(z-z_0)^{-1} + A_2(z-z_0)^{-2} + \dots,$$

let

$$r = \limsup_{n\to\infty} \sqrt[n]{|A_n|}$$

and let γ be the circle $\gamma : |z-z_0| = r$ with interior $I(\gamma)$ and exterior $E(\gamma)$. Then there are three possibilities:

(1) *If $r = 0$, the series is absolutely convergent for all $z \in \overline{\mathbb{C}}$ in the extended complex plain, except for $z = z_0$.*
(2) *If $0 < r < \infty$, the series is absolutely convergent $\forall z \in E(\gamma)$ and divergent $\forall z \in I(\gamma)$.*
(3) *If $r = \infty$, the series diverges $\forall z \neq \infty$.*

Proof. Substitute $\zeta = (z-z_0)^{-1}$ and use Cauchy–Hadamard Theorem 38. □

Note that the convergence is uniform on compact subsets of $E(\gamma)$ by Weierstrass M-test, so by Weierstrass Theorem 39, the sum $f(z)$ of the series above is analytic in $E(\gamma)$. Also, if r is finite, $f(z)$ is analytic at ∞ because $f(1/\zeta)$ is analytic at 0.

Definition 16. *Series*

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

is called a Laurent series. *By definition, such a series converges if and only if its positive and negative parts*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

both converge. The sum of a Laurent series is the sum of sums of the two above series.
Equivalently,

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = S$$

if and only if

$$\lim_{\substack{\mu\to\infty\\ \nu\to\infty}} \sum_{n=-\mu}^{\nu} a_n(z-z_0)^n = S.$$

Immediately from definition, Laurent series converges in an annulus (a ring-shaped region) $D : r < |z-z_0| < R$, where

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad r = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}.$$

Note that convergence is uniform on any compact subset of D (since convergence of positive and negative part is uniform on compact subsets of D). From this point on, assume D is not empty, that is $r < R$.

Theorem 50. *The sum of the Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is analytic on the annulus $D : r < |z - z_0| < R$, and the coefficients a_k are given by formula

$$a_k = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}} dz, \qquad k \in \mathbb{Z}$$

where γ_ρ is any circle $|z - z_0| = \rho$, $r < \rho < R$.

Proof. Analyticity follows immediately from Weierstrass Theorem 39.

From the equality

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

we get

$$\frac{1}{2\pi i} \frac{f(z)}{(z - z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} a_n (z - z_0)^{n-k-1}.$$

Since convergence of the series is uniform on γ_ρ, it can be integrated term-wise. Therefore,

$$\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} a_n \int_{\gamma_\rho} (z - z_0)^{n-k-1} dz.$$

In the latter sum, only one term is nonzero, the one with $n - k - 1 = -1$. We get

$$\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}} = \frac{1}{2\pi i} a_k \cdot 2\pi i,$$

as required. $\square$

Corollary 8. *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

in an annulus D, and

$$\varphi(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

in an annulus Δ, and let $f(z) = \varphi(z)$ for all z on a circle $\gamma : |z - z_0| = \rho$ that belongs both to D and Δ. Then $a_n = b_n$ for every $n \in \mathbb{Z}$.

Theorem 51. (Laurent series expansion) *Let $f(z)$ be an analytic function on the annulus $D : r < |z - z_0| < R$. Then there exits a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converging to $f(z)$ on D. Coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad \forall n \in \mathbb{Z},$$

where γ_ρ is any circle $\gamma_\rho : |z - z_0| = \rho$ with $r < \rho < R$.

Proof. The proof is very similar to the proof of Thereom 37.

Pick numbers r', R' so that $0 < r < r' < R' < R$ and consider annulus $D' : 0 < r' < |z - z_0| < R'$. Let $z_1 \in D'$. Then

$$\frac{f(z)}{z - z_1}$$

is analytic in $D \setminus \{z_1\}$. By Cauchy Integral Theorem for a system of contours, we have

$$\frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}dz = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z - z_1}dz + \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_1}dz,$$

where $\gamma_{R'} : |z - z_0| = R'$, $\gamma_{r'} : |z - z_0| = r'$, and Γ is a circle in D' centered at z_1. Note that the latter integral is equal to $f(z_1)$ by Cauchy Integral Formula. Therefore we have

$$f(z_1) = \frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}dz + \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z_1 - z}dz = I_1 + I_2.$$

We will show that the first integral corresponds to the positive part of a Laurent series, and the second one the the negative part.

Deal with I_1 first. Note that

$$\begin{aligned}
\frac{1}{z - z_1} &= \frac{1}{(z - z_0) - (z_1 - z_0)} = \frac{1}{z - z_0}\frac{1}{1 - \frac{z_1 - z_0}{z - z_0}} = \\
&= \frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{z_1 - z_0}{z - z_0}\right)^n = \\
&= \sum_{n=0}^{\infty}\frac{(z_1 - z_0)^n}{(z - z_0)^{n+1}}.
\end{aligned}$$

Multiplying by $\frac{1}{2\pi i}f(z)$, we get

$$\frac{1}{2\pi i}\frac{f(z)}{z - z_1} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{f(z)}{(z - z_0)^{n+1}}(z_1 - z_0)^n.$$

This series is uniformly convergent on $\gamma_{R'}$ (because $|z - z_0| < |z_1 - z_0|$), so we can integrate term by term along this circle. Have

$$I_1 = \frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}dz = \sum_{n=0}^{\infty} a_n(z_1 - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{(z - z_0)^{n+1}}dz.$$

For I_2, we perform a similar calculation:

$$\frac{1}{z_1 - z} = \frac{1}{(z_1 - z_0) - (z - z_0)} = \sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(z_1 - z_0)^{n+1}},$$

and

$$\frac{1}{2\pi i}\frac{f(z)}{z_1 - z} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{f(z)}{(z - z_0)^{-n}}(z_1 - z_0)^{-n-1} = \sum_{n=1}^{\infty}\frac{1}{2\pi i}\frac{f(z)}{(z - z_0)^{-n+1}}(z_1 - z_0)^{-n}$$

is uniformly convergent on $\gamma_{r'}$ (because $|z_1 - z_0| < |z - z_0|$). Therefore,

$$I_2 = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z_1 - z}dz = \sum_{n=1}^{\infty} a_{-n}(z_1 - z_0)^{-n},$$

where

$$a_{-n} = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{(z - z_0)^{-n+1}}dz.$$

Putting I_1 and I_2 together, we get

$$f(z_1) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}a_n(z_1 - z_0)^n + \sum_{n=1}^{\infty} \frac{1}{2\pi i}a_{-n}(z_1 - z_0)^{-n} = \sum_{n=-\infty}^{\infty} a_n(z_1 - z_0)^n.$$

Since $z_1 \in D$ is arbitrary, this is precisely what's required. To prove the asserted equality for a_n, it is only left to note that by Cauchy's Integral Theorem, the integrals representing a_n do not change their values if we change the path of integration to γ_ρ. □

EXAMPLES.

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}, \quad r = 0,\ R = \infty.$$

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\sum_{n=-1}^{\infty} z^n, \quad 0 < |z| < 1.$$

$$\frac{1}{z(z-1)} = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=-\infty}^{-2} z^n, \quad 1 < |z| < \infty.$$

Note that the latter two are examples of *different* Laurent series for the same function $\frac{1}{z(z-1)}$ with the same center $z_0 = 0$. However, there is no contradiction with uniqueness of Laurent series (Corollary 8), because the corresponding annuli *do not intersect*.

Corollary 9. (Cauchy's inequalities) *Let f be analytic in $r < |z - z_0| < R$, and $r < \rho < R$, and $\gamma_\rho : |z - z_0| = \rho$. Let $M(\rho) = \max_{\gamma_\rho} |f(z)|$. If a_n are the coefficients of the Laurent series for f in the annulus $r < |z - z_0| < R$, then*

$$|a_n| \le \frac{M(\rho)}{\rho^n}$$

for all $n \in \mathbb{Z}$.

Proof. This is a direct corollary of Theorem 51 and the formula for a_n in that theorem. □

Another observation is that the reasoning of Section 11.1 applies to the inner and outer radii of the annulus of convergence of a Laurent series. With natural adjustments, the proof of Theorem 48 works to obtain the following statement.

Theorem 52. *A function given by a Laurent series with annulus of convergence $0 < r < |z - z_0| < R < \infty$ has at least one singular point on the circle $|z - z_0| = R$ and at least one singular point on the circle $|z - z_0| = r$.*

If $f(z)=\sum_{\mathbb{Z}} a_n(z-z_0)^n$ is a Laurent expansion of f, we say that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the *regular* part of the Laurent series, and $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ is the *principal* part of the Laurent series.

If for some function f it happens that $f(z)=\sum_{\mathbb{Z}} a_n(z-z_0)^n$ with $0<|z-z_0|<R$, we say that $\sum_{\mathbb{Z}} a_n(z-z_0)^n$ is a Laurent expansion of f at z_0. If the $f(z)=\sum_{\mathbb{Z}} a_n z^n$ in an annulus $r<|z|<\infty$, we say that $\sum_{\mathbb{Z}} a_n z^n$ is a Laurent expansion of f at infinity. In the latter case $\sum_{n=1}^{\infty} c_n(z-z_0)^n$ is the principal part of Laurent series at infinity, and $\sum_{n=0}^{\infty} a_{-n}(z-z_0)^{-n}$ is the regular part of Laurent series at infinity.

Lecture 12. Isolated singular points. Meromorphic functions. Residue theorem

April 19, 2017
Relevant Sections in Markushevich:
II.1.1–4, II.10.50, II.2.6

12.1. **Isolated singular points.** Observe that a point can be a singular point of a function for different reasons. For example, observe that 0 is a singular point of $\sqrt{z}$ and of $\frac{1}{z}$. In this section we look into a specific kind of singular points, defined below.

Definition 17. *We say that a point a is an* isolated singular point *of a function $f(z)$ if $f(z)$ is not defined analytically at $z=a$, but is analytic in a punctured neighborhood of a.*

There is some awkwardness in terminology:

- According to this definition, an isolated singular point is not necessarily a singular point in the sense of the preceding section, since f being not defined at a is not the same as it being impossible to analytically extend f to a. See also case R of the next definition.
- Another quirk of this term is that the term itself suggests that if a singular point is not an isolated singular point, then there must be other singular points close to it. This is not necessarily the case. For example, 0 is a singular point, but not an isolated singular point, of any branch of $\sqrt{z}$, since $\sqrt{z}$ cannot be defined analytically on a punctured neighborhood of 0.

Since this terminology is established and widely used, we will have to put up with this.

Definition 18. *Suppose a is an isolated singular point of f. Expand f in a Laurent series at $z=a$. There are three options:*

R *Principal part is zero: $c_n=0$ for $n<0$. In this case a is called a* removable singular point.

P *Principal part is nonzero but contains only finitely many nonzero terms: $c_n=0$ for $n<-N$. In this case a is called a* pole*, and the largest power of $1/(z-a)$ is called the* order of a pole. *(In other words, order of a pole is a number N such that $c_n=0$ for $n<-N$ and $c_{-N}\neq 0$.) A pole of order* 1 *is called* simple.

E *Principal part contains infinitely many nonzero terms. In this case we say that a is an* essential singular point.

REMARK. This definition, in particular, works for Laurent series at infinity, in which case the principal part is the sum of positive powers (sign on n should be reversed in the formulas above in that case).

Theorem 53. (Classification of isolated singular points) *Let a be an isolated singular point of a function $f(z)$. Consider the limit*

$$\lim_{z\to a} f(z).$$

Then there is the following relation between type of singularity at $z = a$ and value of this limit:

R $\Leftrightarrow$ *the limit is finite (or, equivalently, $f(z)$ is bounded in a neighborhood of point a),*
P $\Leftrightarrow$ *the limit is infinite,*
E $\Leftrightarrow$ *the limit does not exist.*

Proof. Case R:
If f is bounded in a punctured neighborhood of a, (which is true if a finite limit at a exists), then by Cauchy's Inequalities (Corollary 9), $|c_{-n}| \le M/\rho^{-n}$. Taking limit $\rho \to 0$, get that $c_{-n} = 0$.

Other direction is obvious: if $c_{-n} = 0$, then function is represented by a Taylor series.

Case P:
Let $\lim_{z\to a} f(z) = \infty$. Put $g(z) = 1/f(z)$. Then $\lim_{z\to a} g(z) = 0$. By the previous case, g is bounded in some punctured neighborhood of a, so g expands in a Taylor series at a with first nonzero coefficient c_k, $k \ge 1$:

$$g(z) = c_k(z-a)^k + c_{k+1}(z-a)^{k+1} + \ldots = c_k(z-a)^k(1+\varphi(z)),$$

where $\varphi(a) = 0$, so $1 + \varphi(z) \ne 0$ on a neighborhood of a. Then $\frac{1}{1+\varphi(z)}$ is analytic in a neighborhood of a, expands in a Taylor series at a, and is nonzero at a. Then

$$f(z) = \frac{1}{c_k(z-a)^k} \cdot \frac{1}{1+\varphi(z)} = \frac{1}{c_k(z-a)^k} \sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=-k}^{\infty} \tfrac{a_{n+k}}{c_k}(z-a)^n,$$

which exactly means that f has a pole of order k at a.

Other direction: $f(z) = \sum_{n=1}^{k} c_{-n}(z-a)^{-n} + \sum_{n=0}^{\infty} c_n(z-a)^n$. The latter sum is analytic, while the former sum $\to \infty$ as $z \to a$.

Case E:
There are no other possibilities left for either direction of statement. $\square$

EXAMPLES. All of the following (when the points are actually isolated singular) can be established by either inspecting the corresponding limits, or by finding principal part of Laurent series.

(1) $f(z) = 1/z^5$ has an order 5 pole at 0 and is analytic (has removable singularity) at ∞ since $f(1/z) = z^5$, which is analytic at 0 (or since ∞ is an isolated singular point and the limit of $\frac{1}{z^5}$ at ∞ is finite).
(2) A polynomial of degree n has a pole of order n at infinity.
(3) $e^{1/z}$ has an essential singular point at 0 since this function is analytic on $\mathbb{C}$ and

$$\lim_{x\to 0+} e^{1/x} = \infty, \quad \lim_{x\to 0-} e^{1/x} = 0.$$

One can also readily see that by looking at Laurent series of $e^{1/z}$.

(4) $1/\cos(1/z)$ does not have an isolated singular point at 0 since $\cos(1/z)$ has zeros in any neighborhood of 0. NOTE that one can show that $\lim_{z\to 0}$ does not exist, but it *does not* imply that we have an essential point, because to apply Theorem 53 we need to have an isolated singular point in the first place.

At $z_k = 1/(\frac{\pi}{2} + \pi k)$, $1/\cos(1/z)$ has simple poles (why?). Finally, at ∞ (notice that $1/\cos(1/z)$ is analytic for $|z| > 2/\pi$) the singularity is removable since $\lim_{z\to\infty} 1/\cos(1/z) = 1/\cos 0 = 1$.

(5) $f(z) = \sqrt{z}$ does not have an isolated singular point at 0 since $\sqrt{z}$ is not defined in a punctured neighborhood of 0. NOTE that $\lim_{z\to 0} f(z) = 0$, but it *does not* imply that we have a removable singularity, because to apply Theorem 53 we need to have an isolated singular point in the first place.

(6) $f(z) = 1/\sqrt{z}$ does not have an isolated singular point at 0 since $\sqrt{z}$ is not defined in a punctured neighborhood of 0. NOTE that $\lim_{z\to 0} f(z) = \infty$, but it *does not* imply that we have a pole, because to apply Theorem 53 we need to have an isolated singular point in the first place.

(7) $\tan z$ has simple poles at zeros of $\cos z$, that is at $z \in \{\frac{\pi}{2} + \pi k\}$. Infinity is not an isolated singular point since there are singular points in any neighborhood of infinity.

For essential singular points there is a stronger statement.

Theorem 54. (Casorati–Weierstrass, or Sokhotski Theorem) *If a is an essential singular point of a function f, then for every $A \in \overline{\mathbb{C}}$ there is a sequence $z_n \to a$ such that $f(z_n) \to A$ as $n \to \infty$.*

Proof. Suppose this statement fails for $A = \infty$. Then f is bounded in a neighborhood of a, so a is in fact a removable singular point.

Suppose the statement fails for a finite A, which means that there is a neighborhood B of A such that values $f(z)$ miss B for all z in any, however small, neighborhood of a. Therefore a function $g(z) = \frac{1}{f(z)-A}$ is analytic in a neighborhood of a, so a is a removable singular point for g. But then since $f(z) = A + \frac{1}{g(z)}$, a is either removable singular point of f, or a pole of f. $\square$

REMARK. In fact, even stronger statement is true (Picard's Theorem): in any neighborhood of an essential singular point, f must take all values except, perhaps, one.

EXAMPLE. Consider the function $f(z) = \sin\frac{1}{z}$. This function has an essential singular point at 0. Given an $A \in \mathbb{C}$, find w_0 such that $\sin(w_0) = A$ (sin is surjective). Then for $z_k = 1/(w_0 + 2\pi k)$, $\sin(1/z_k) = A$. Therefore, given an arbitrary $A \in \mathbb{C}$ we found a sequence $z_k \to 0$ s.t. the sequence of values of f not only approaches A, but is actually constant and equal to A. Also note that by considering $g(z) = \sin z$ we can organize the same behavior at ∞.

12.2. **Meromorphic functions.** Note that if a function has a pole at point a, it is relatively "tame" on a neighborhood of that point, since in such case

$$f(z) = c_{-k}(z-a)^{-k} + \ldots + c_{-1}(z-a)^{-1} + c_0 + c_1(z-a) + \ldots = (z-a)^{-k}\varphi(z),$$

where φ is analytic at a. With that in mind, we give the following definition.

Definition 19. *Let G be a domain. If a function f is analytic in G except a finite number of points, all of which are poles or removable singular points, we say that f is* meromorphic *in G.*

Example 1. Any rational function is meromorphic on any domain. (Recall that rational function is a ratio of two polynomials.)

Example 2. The function $f(z) = 1/\sin z$ has simple (order 1) poles at zeros of $\sin z$, so f is meromorphic on any bounded domain, but not on $\mathbb{C}$ (infinitely many poles) or $\overline{\mathbb{C}}$ (infinitely many poles, and ∞ is a singular point that's not a pole, in fact, not an isolated singularity).

Remark. Sometimes (in fact, in most textbooks) the requirement that there are only finitely many poles is omitted in the definition of meromorphic function. The resulting notion is close since by Uniqueness Theorem any compact subset of the domain will contain only finitely many poles. Within such terminology, the above function $1/\sin z$ is meromorphic on $\mathbb{C}$ but not on $\overline{\mathbb{C}}$.

Note that if $G = \overline{\mathbb{C}}$, then the words "finite number" can be omitted from the definition, because if a function has infinitely many poles, then by compactness of $\overline{\mathbb{C}}$ the set of poles has a limit point in $\overline{\mathbb{C}}$, making it not a point of analyticity and not a pole.

Theorem 55. *Suppose f is an entire function. Then infinity is a removable singular point for f if and only if $f = \mathrm{const}$, and is a pole of order n if and only if f is a polynomial of degree n.*

Proof. If ∞ is a removable singularity, then by classification theorem f is bounded and therefore f is constant by Liouville's Theorem. Other direction is clear.

If ∞ is a pole of degree n, that means the principal part of the Laurent expansion at infinity is a polynomial of degree n. Since f is analytic, there are no negative powers in Laurent expansion. □

With the above theorem in mind, we can say that Liouville's theorem states that a function analytic on $\overline{\mathbb{C}}$ must be constant. Being meromorphic on $\overline{\mathbb{C}}$ also turns out to be a strong constraint, as we see in the following statement.

Theorem 56. *A function $f(z)$ is meromorphic in $\overline{\mathbb{C}}$ if and only if f is a rational function.*

Proof. Suppose f is meromorphic in $\overline{\mathbb{C}}$ and $a_1, a_2, \ldots, a_n$ are the poles of f. Expand f in a Laurent series at each a_k. At each a_k, the principal part of Laurent series is a polynomial $P_k\left(\frac{1}{z-a_k}\right)$. Then the function

$$f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z-a_k}\right)$$

is entire, and therefore is a polynomial $P(z)$ by the previous theorem, so

$$f(z) = P(z) + \sum_{k=1}^{n} P_k\left(\frac{1}{z-a_k}\right), \tag{16}$$

which is a rational function. □

Remark. The polynomial $P(z)$ above, up to a constant, is just the principal part of the Laurent series of f at ∞.

Remark. Observe that the equation 16 is nothing other than decomposition of f into partial fractions.

Theorem 57. *Any bijective meromorphic*[4] *mapping* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is a Möbius transformation.*

Proof. We have two cases, whether $\infty = f^{-1}(\infty)$ or not.

Case 1. $\infty = f^{-1}(\infty)$. This means that f is analytic on $\mathbb{C}$ and has a pole at ∞. By Theorem 55, f is polynomial. By Fundamental Theorem of Algebra (Corollary 4), the equation $f(z) = z_0$, where $f'(z_0) \neq 0$, has $\deg f$ distinct roots, so in order for f to be injective $\deg f$ must be 1, i.e. $f(z) = az + b$, which is a Möbius transformation.

Case 2. $\infty \neq f^{-1}(\infty) = A$. This means that f has a removable singularity at ∞ (because by bijectivity, $f(\infty)$ must be finite), and a pole at A (because $f(A) = \infty$, so it satisfies case P of Classification theorem 53). Therefore, f is a polynomial of $\frac{1}{z-A}$ (see proof of Theorem 56). Same as above, this polynomial must be of degree 1 to allow injectivity, so $f(z) = \frac{a}{z-A} + b$, which is a Möbius transformation. □

12.3. **Residue theorem.** Suppose a function f is analytic in G except for finitely many points $a_1, \dots, a_n$. Suppose a closed simple rectifiable curve γ is such that $\gamma \cup I(\gamma) \subseteq G$, and $a_1, \dots, a_n \in I(\gamma)$. Then the points a_k "get in the way" of applying Cauchy Theorem to $\int_\gamma f$. But we can put a small circle γ_k around each point γ_k, then by Cauchy Theorem for a system of contours we have

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz.$$

Note that if $\sum_{\mathbb{Z}} c_n(z-a_k)^n$ is a Laurent expansion of f at a_k, then we can integrate f termwise:

$$\int_{\gamma_k} f(z)dz = 2\pi i c_{-1}.$$

Definition 20. *The Laurent coefficient c_{-1} in the Laurent expansion of f at an isolated singular point a is called the* residue of a function f *at the point a, denoted* $\operatorname*{res}_a f$.

If a is a regular (or removable singular) point of f, $\operatorname*{res}_a f = 0$.

Theorem 58. (Residue Theorem) *Suppose G is a domain, γ is a closed simple rectifiable curve γ is such that $\gamma \cup I(\gamma) \subseteq G$. Suppose f is analytic in G, except for points $a_1, a_2, \dots, a_n \in I(\gamma)$. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{a \in I(\gamma)} \operatorname*{res}_a f = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{a_k} f.$$

Proof. By Cauchy Integral Theorem for a system of contours, switch to integrating along small circles γ_k around the points a_k, then use the formula $\int_{\gamma_k} f(z)dz = 2\pi i c_{-1}$ above, and the definition of a residue. □

[4] If we view $\overline{\mathbb{C}}$ as a Riemann sphere, we can say "analytic" instead of meromorphic, which sounds prettier.

REMARK. The statement also works for a system of contours.

REMARK. If we use stronger version of Cauchy's Theorem (theorem 16), we can generalize this statement to $\gamma = \partial G$, where G is a bounded domain and f analytic in G and continuous in $\overline{G}$, except for finite number of points.

Lecture 13. RESIDUES AND THEIR APPLICATIONS. ELEMENTARY MULTI-VALUED FUNCTIONS

April 26, 2017
Relevant Sections in Markushevich:
II.2.6-7, I.11.53-54,56.

13.1. Computing residues.

(1) If a is a pole of order 1, then

$$f(z) = c_{-1}(z-a)^{-1} + c_0 + c_1(z-a) + \dots$$

so

$$\operatorname*{res}_a f = c_{-1} = \lim_{z\to a}(z-a)f(z).$$

In particular, if $f(z) = \frac{g(z)}{h(z)}$, where h has a simple zero at $z = a$, that is $h(a) = 0$, $h'(a) \neq 0$, then

$$\begin{aligned}\operatorname*{res}_a f &= \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{(z-a)g(z)}{h(z)} = \\ &= \lim_{z\to a}\frac{g(z) + (z-a)g'(z)}{h'(z)} = \\ &= \frac{g(a)}{h'(a)}\end{aligned}$$

by L'Hospital's rule (follows by writing out Taylor series in $z - a$ in the numerator and denominator).

EXAMPLE. Keeping the above in mind sometime saves time:

$$\operatorname*{res}_i \frac{1}{z^{2016}-1} = \lim_{z\to i}\frac{z-i}{z^{2016}-1} = \lim_{z\to i}\frac{(z-i)'}{(z^{2016}-1)'} = \lim_{z\to i}\frac{1}{2016z^{2015}} = \frac{i}{2016}.$$

(2) If a is a pole of order n, then

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \dots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \dots,$$

so

$$(z-a)^n f(z) = c_{-n} + \dots + c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \dots,$$

and

$$\frac{d^{n-1}}{dz^{n-1}}(f(z)(z-a)^n) = (n-1)!c_{-1} + \dots$$

thefore

$$\operatorname*{res}_a f = c_{-1} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(f(z)(z-a)^n).$$

(3) If a is an essential singular point, then to find a residue at a, one needs to expand f in a Laurent series and find c_{-1} explicitely, or to integrate along a closed curve encircling a (which defeats the purpose of notion of residue, but still).

Example.
$$\sin\left(\frac{1}{z}\right)=\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\dots,$$
so
$$\operatorname*{res}_0 \sin\left(\frac{1}{z}\right)=1.$$

13.2. **Using residues to compute integrals.** Residues, of course, can be employed straightforwardly to compute integrals.

Example.
$$\operatorname*{res}_i \frac{\sin z}{z^2+1}=\frac{\sin i}{2i},\quad \operatorname*{res}_{-i} \frac{\sin z}{z^2+1}=\frac{\sin(-i)}{2(-i)}=\frac{\sin i}{2i},$$
so
$$\int_{|z|=2}\frac{\sin z}{z^2+1}dz=2\pi i\left(\operatorname*{res}_i \frac{\sin z}{z^2+1}+\operatorname*{res}_{-i} \frac{\sin z}{z^2+1}\right)=2\pi\sin i.$$

13.2.1. *Using residues to compute real integrals.* More interestingly, residues can be used to find real line integrals. There are quite a few classes of integrals that can be computed by certain standard tricks. They are best found in undergraduate complex analysis books. Below we list just a few of examples, without going too much into details.

(1) Suppose P and Q are polynomials such that $\deg P\le \deg Q-2$, and $Q(x)\ne 0$ for $x\in\mathbb{R}$. The the following improper integral converges:
$$I=\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx.$$
We can compute this integral using residue theorem. Consider contour L_R that consists of a segment of real line $[-R,R]$ and upper semicircle γ_R of radius R. Then
$$\int_{L_R}\frac{P(z)}{Q(z)}dz=\int_{-R}^{R}\frac{P(z)}{Q(z)}dz+\int_{\gamma_R}\frac{P(z)}{Q(z)}dz.$$
Note that the second integral goes to 0 as $R\to\infty$:
$$\left|\int_{\gamma_R}\frac{P(z)}{Q(z)}dz\right|\le M\cdot\frac{1}{R^2}\cdot\pi R\to 0.$$
Therefore, taking $R\to\infty$ (in particular, semicircle of radius R then contains all zeros of Q in upper half-plane), we have
$$I=2\pi i\sum_{\operatorname{Im} a_k>0}\operatorname*{res}_{a_k}\frac{P}{Q},$$
where a_k are all zeros of Q, so the summation is over all zeros of Q *in the upper halfplane*.

(2) Integrals of the form
$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos\alpha x\,dx,\quad \int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin\alpha x\,dx,$$
where $\deg P\le\deg Q-1$, $Q(x)\ne 0$ on $\mathbb{R}$, and $\alpha>0$, can be computed by considering integral of the function
$$f(z)=\frac{P(z)}{Q(z)}e^{i\alpha z}$$

along the same contour L_R as in the previous item:

$$\int_{L_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz = \int_{-R}^{R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz + \int_{\gamma_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz.$$

The integral along γ_R goes to 0 ($R \to \infty$), but it is not as immediate as in the previous case. In fact, it's a whole named statement, Jordan's Lemma. We don't prove it or even state it here for lack of time. Taking it for granted, we can take real and imaginary part of the limit ($R \to \infty$) of the above equality to see that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x \, dx = -2\pi \sum_{\operatorname{Im} a_k > 0} \operatorname{Im}(\operatorname*{res}_{a_k} f)$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x \, dx = 2\pi \sum_{\operatorname{Im} a_k > 0} \operatorname{Re}(\operatorname*{res}_{a_k} f),$$

where f is defined above.

(3) We will properly define the power function z^α later in this lecture. For now, let's take it for granted. Then we can compute the integral

$$\int_0^\infty \frac{x^\alpha P(x) dx}{Q(x)},$$

where $0 < \alpha < 1$, $\deg P \leq \deg Q - 2$, $Q(x) \neq 0$ for $x > 0$, and $Q(x)$ has a zero of order at most 1 at the origin (order of zero will be also introduced later in this lecture).

Consider the function $f(z) = \frac{z^\alpha P(z)}{Q(z)}$ and the "incised circle" contour that goes from εi to $R + \varepsilon i$ in a horizontal straight line, then makes almost a full circle from $R + \varepsilon i$ to $R - \varepsilon i$, then straight line to $-\varepsilon i$, then clockwise semicircle back to εi. Taking limit as $\varepsilon \to 0$ and $R \to \infty$, we can see that

$$\int_0^\infty \frac{x^\alpha P(x) dx}{Q(x)} = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \sum_{a_k \neq 0} \operatorname*{res}_{a_k} f,$$

where f is defined above, and the sum is taken over all residues at nonzero poles.

(4) We can also compute integrals of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta,$$

where F is a reasonable function (for example, any rational function). To do that, we notice that if $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, and $dz = ie^{i\theta} d\theta$, so $d\theta = \frac{dz}{iz}$. Finally, notice that, for $0 \leq \theta \leq 2\pi$, the point $z = e^{i\theta}$ traverses the unit circle γ, so we have

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta = \int_\gamma F\left(\tfrac{1}{2}(z + \tfrac{1}{z}), \tfrac{1}{2i}(z - \tfrac{1}{z})\right) \frac{dz}{iz} = 2\pi i \sum_{|a_k| < 1} \operatorname*{res}_{a_k} f,$$

where $f(z) = \frac{F\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)}{iz}$, and the summation is over all residues inside the unit circle.

13.2.2. *Using residues to find sum of series.* Another amusing application of residue calculus is finding sums like

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.$$

The idea is to organize a function with residues $\frac{1}{n^2}$ and look into its integral along an appropriate contour. This is done in detail in Homework.

13.3. **Multiplicity of zeros and poles. Relation between number of zeros and number of poles.** There is a notion "twin" to the order of a pole.

Definition 21. *Suppose f is analytic at a. Then we say that a is a zero of order k if $f(a) = f'(a) = \ldots = f^{(k-1)}(a) = 0$ and $f^{(k)}(a) \neq 0$. Equivalently, $f(z) = (z-a)^k \varphi(z)$ with φ analytic at a and $\varphi(a) \neq 0$. If $f = 0$, we say that multiplicity is infinite.*

Recall that a is a pole of order k if $f(z) = \frac{\varphi(z)}{(z-a)^k}$, where φ is analytic at a (recall that this means "analytic on a neighborhood of a") and $\varphi(a) \neq 0$.

In other words, if a is a regular point or a pole of f, we can find a number m such that $f(z) = (z-a)^m \varphi(z)$, where φ is analytic at a and $\varphi(a) \neq 0$. Then the number $\operatorname{ord}_a f = m$ is called *order* of point a (with respect to a function f).

It is easy to see that $\operatorname{ord}_a(fg) = \operatorname{ord}_a f + \operatorname{ord}_a g$.

Suppose f is analytic in a neighborhood of a point a, and a is a regular point or a pole of f.

Lemma 7. *If a is a regular point or a pole of f, then*

$$\operatorname*{res}_a \left(\frac{f'}{f} \right) = \operatorname{ord}_a f.$$

Proof. Let $n = \operatorname{ord}_a f$. Then $f(z) = (z-a)^n \varphi(z)$, where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$. Then $f'(z) = n(z-a)^{(n-1)}\varphi(z) + (z-a)^n \varphi'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}\varphi(z) + (z-a)^n \varphi'(z)}{(z-a)^n \varphi(z)} = \frac{n}{z-a} + \frac{\varphi'(z)}{\varphi(z)}.$$

Function $\frac{\varphi'(z)}{\varphi(z)}$ is analytic at a, so $\operatorname*{res}_a \frac{f'}{f} = c_{-1} = n$. ☐

Theorem 59. *Let G be a domain, γ a simple rectifiable closed curve contained in G together with its interior, $\gamma \cup I(\gamma) \subseteq G$, and let a function f be meromorphic in G without zeros or poles on γ. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P,$$

where N is the number of zeros of f in $I(\gamma)$ and P is the number of poles of f in $I(\gamma)$ (counting multiplicity).

Proof. By Residue Theorem 58, we get that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{a_k \in I(\gamma)} \operatorname*{res}_{a_k} \frac{f'}{f}.$$

By the Lemma above, the latter sum is just $N - P$, since a zero of multiplicity n contributes n to this sum, and a pole of multiplicity n contributes $-n$. ☐

REMARK. Just like Residue theorem, this also holds for γ that is a boundary of a bounded domain, or for a γ that is a system of contours.

The expression $\frac{f'}{f}$ is called a *logarithmic derivative* of f, since "$(\ln f(z))' = \frac{f'(z)}{f(z)}$." This gives us another approach to compute the integral in the above theorem, by using the Fundamental Theorem of Calculus. However, that will involve the multivalued function Ln, so to proceed with that plan we first need to properly explain how to deal with multivalued functions.

13.4. **Elementary multi-valued functions.** This sections is somewhat of a crutch to avoid dealing with the general construction. But we actually do not need arbitrary multi-valued functions here, but rather just a few of them.

Suppose G is a domain in $\mathbb{C}$, and $f : G \to \mathbb{C}$ is a function for which we want to consider an inverse $f^{-1} : f(G) \to G$. The problem is that f is not assumed to be injective, so f^{-1} may not be defined a (single-valued) function.

Definition 22. *A (single-valued) function $F : A \to B$ is a subset $F \subseteq A \times B$ such that*

(1) *For any $a \in A$, there is $b \in B$ s.t. $(a, b) \in F$. Notation: $F(a) = b$.*
(2) *For any $a \in A$, if $(a, b_1) \in F$ and $(a, b_2) \in F$, then $b_1 = b_2$ ("vertical line test").*

A multi-valued function $F : A \to B$ is a subset $F \subseteq A \times B$ such that

(1) *For any $a \in A$, there is $b \in B$ s.t. $(a, b) \in F$. Notation: $F(a) = b$.*

Other way to give the same definition is to say that a multi-valued function $F : A \to B$ is a (single-valued) function from A to the set of nonempty subsets of B.

Return now to our case, $f : G \to \mathbb{C}$. If function f is not injective, the function f^{-1} is multi-valued. Below we construct so-called single-valued branches of f^{-1}.

Suppose (!) that there are countably many subdomains $G_1, G_2, \dots$ of G so that

- G_k's are disjoint, i.e. $G_k \cap G_{k'} = \emptyset$ for $k \neq k'$,
- restriction $f|_{G_k}$ is injective for each k,
- every point $z \in G$ is either a point of some G_k, or is a point of common boundary of at least two distinct G_k's: $z \in G_k$ or $z \in \partial G_k \cap \partial G_{k'}$, $k \neq k'$.

Let E denote the union of the points of G that lie in common boundary of at least two subdomains G_k. Then G decomposes in a disjoint union:

$$G = E \cup G_1 \cup G_2 \cup \dots$$

Then, since f is injective on G_k, it has a single-valued inverse $(f|_{G_k})^{-1} : f(G_k) \to G_k \subseteq G$, which we denote by f_k^{-1}. Each function f_k^{-1} is called a *single-valued branch* of f^{-1}.

Note that there is no guarantee that such decomposition exists. One can prove that it exists if f is analytic, but that's outside this course. We, however, are only going to need this for few specific functions, so we need not worry about the general case.

EXAMPLE. Consider $f(z) = z^n$, n a positive integer. Then one can easily give an appropriate decomposition, for example, one shown in the Figure 4: the full angle at the origin is divided into n equal angles.

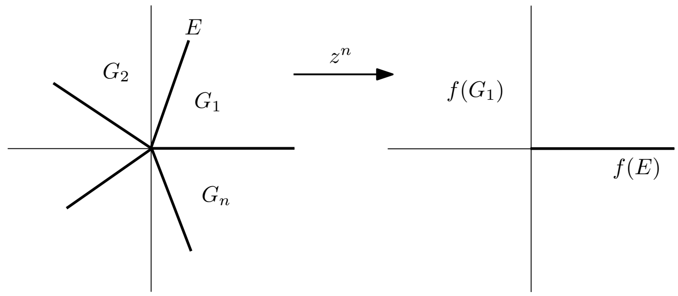

FIGURE 4. On the left, the bolded part is the set E. On the right, $f(G_k) = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

In this example, $G_k = \{z \in \mathbb{C} \mid 2\pi(k-1)/n < \arg z < 2\pi k/n\}$. For instance, taking $k = 1$ we get an injective function $z^n|_{G_1}$ whose inverse $\sqrt[n]{z}$ is defined on the domain $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and valued in $G_1 = \{z \in \mathbb{C} \mid 0 < \arg z < 2\pi/n\}$.

Note that the domain $f(G_k)$ in the Fig. 4 is not all that good: it misses all non-negative reals. So the branches of nth root that correspond to the decomposition in that figure are not defined on non-negative real numbers, which is really inconvenient. But this is not actually a problem, because we can just pick a different decomposition $G = E \cup G_1 \cup G_2 \cup \dots$ (see Fig. 5)

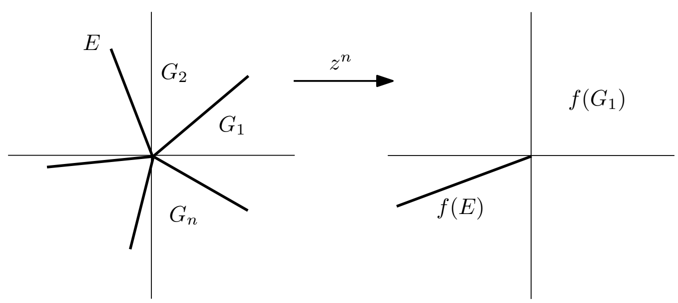

FIGURE 5. On the left, the bolded part is the set E. On the right, $f(G_k) = \mathbb{C} \setminus f(E)$.

A decomposition $G = E \cup G_1 \cup G_2 \cup \dots$ is often implied when dealing with multi-valued functions. For example, typical phrase in a book would be "Let $\sqrt{x}$ be a branch of square root on the domain $D = \{\operatorname{Re} z > 0\}$ such that $\sqrt{4} = 2$". This means that the decomposition is picked so that $f(G_k)$ contains the domain D (i.e. $f(E)$ misses D), and then a branch is picked so that the corresponding G_k contains the point 2. Sometimes mention of a specific domain D is omitted; in such case, it is implied that D is a neighborhood of the point (in the example, a neighborhood of the point 4). There is ambiguity in the choice of the decomposition $G = E \cup G_1 \cup G_2 \cup \dots$ in either case, but as long as image of some G_k contains D, it gives the same values of the single-valued branch on D. Indeed, given two decompositions $G = E \cup G_1 \cup G_2 \cup \dots = E' \cup G'_1 \cup G'_2 \cup \dots$, if $D \subseteq f(G_1)$ and $D \subseteq f(G'_1)$, then $f^{-1}|_{G_1}$ and $f^{-1}|_{G'_1}$, while being different functions, take the same values on D because those values belong both to G_1 and G'_1.

13.4.1. *The nth root function.* Let $f(z) = z^n$. Then split $\mathbb{C}$ into pieces

$$G_k = \{z \in \mathbb{C} \mid \alpha_0 + 2\pi(k-1)/n < \arg z < \alpha_0 + 2\pi k/n\}, \quad k = 1, 2, \ldots, n,$$

with arbitrary α_0. This gives n single-valued branches of the nth root function, each defined on the set $\mathbb{C} \setminus \{\operatorname{Arg} z = n\alpha_0\}$.

13.4.2. *The complex logarithm function.* Let $f = e^z$. Since e^z is $2\pi i$-periodic, we can cup up the complex plane into countably many horizontal strips of the hight 2π:

$$G_k = \{z \in \mathbb{C} \mid y_0 + 2\pi k < \operatorname{Im} z < y_0 + 2\pi(k+1)\}, \quad k \in \mathbb{Z}.$$

On each G_k the exponential is injective. We get countably many single-valued branches of the complex logarithm, denoted $\operatorname{Ln} z$, each defined on the set $\mathbb{C} \setminus \exp(\{x + iy_0\}) = \mathbb{C} \setminus \{\operatorname{Arg} z = y_0\}$.

13.4.3. *The exponential function with arbitrary base.* Now that we have Ln, we can define a^z for arbitrary nonzero $a \in \mathbb{C}$. Just put

$$a^z = e^{z \operatorname{Ln} a}.$$

One can easily see that each branch of this function satisfies all the usual properties of the exponential function.
EXERCISE. Check that $z^{1/n}$ defined like that is the same function as $\sqrt[n]{z}$ defined in the Subsection 13.4.1

Note that all our machinery applies to single-valued branches of these functions. In particular, they all are analytic on the corresponding domains, and one can easily find their derivatives, Taylor series, etc, using usual calculus techniques. For example, one can see that the Taylor series are as follows.

$$\operatorname{Ln}(1+z) = z - z^2/2 + z^3/3 - z^4/4 + z^5/5 - z^6/6 + \ldots,$$

for a branch s.t. $\operatorname{Ln}(1) = 0$. Radius of convergence is 1 (exercise: why?).

$$(1+z)^\alpha = 1 + \alpha z + \tfrac{\alpha(\alpha-1)}{2} z^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{6} z^3 + \tfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24} z^4 + \ldots,$$

or using $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ notation,

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,$$

for a branch s.t. $1^\alpha = 1$. Radius of convergence is ∞ if α is a non-negative integer, or 1 otherwise (exercise: why?).

13.4.4. *Defining f^{-1} on a curve.* We will need one more particular thing: now we can define values of, for example, $f^{-1}(z) = \operatorname{Ln} z$ along a given curve. Indeed, let $\gamma : [a,b] \to \mathbb{C}$ be a curve that does not pass through 0. Then we can find a continuous image of γ under f^{-1} as follows. Pick arbitrary initial value of f^{-1} at $\gamma(a)$. Then partition $[a,b]$ into intervals $[a_j, a_{j+1}]$ so that the corresponding pieces $\gamma_j : [a_j, a_{j+1}] \to \mathbb{C}$ stay inside some $f(G_k)$, except for maybe $\gamma(a_j) \in f(E)$ and $\gamma(a_{j+1}) \in f(E)$. Assume that each point $\gamma(a_j)$ is either a point of some $f(G_k)$, or a point of common boundary of exactly two f(G_k)'s (as it is the case with Ln). Then as we pass through the points $\gamma(a_j)$ we have a unique choice for a branch of f^{-1} on γ_{j+1} given such choice for γ_j (see Fig. 6).

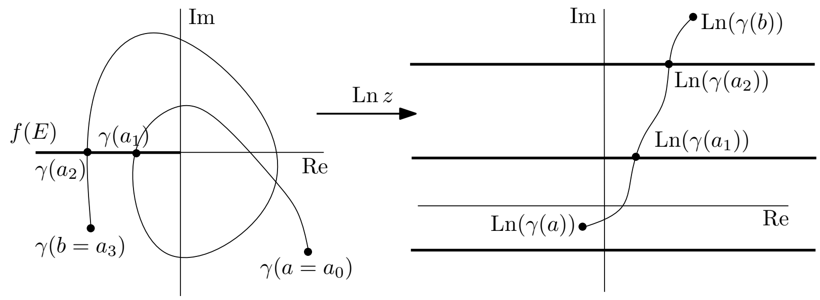

FIGURE 6. Starting with an arbitrary value of Ln at $\gamma(a)$, we can "lift" continuously the whole curve.

Note that even if the curve γ is closed ($\gamma(a) = \gamma(b)$), it does not guarantee that $f^{-1}(\gamma(a)) = f^{-1}(\gamma(b))$. The value $f^{-1}(\gamma(b)) - f^{-1}(\gamma(a))$ is the net change of $g = f^{-1}$ (in our example, of Ln) along γ, denoted

$$\Delta_\gamma(g) = g(\gamma(b)) - g(\gamma(a)).$$

Lecture 14. ARGUMENT PRINCIPLE AND ROUCHÉ THEOREM. BONUS TRACK

May 3, 2017
Relevant Section in Markushevich:
II.2.7.

14.1. **Argument Principle, Rouché Theorem, and Open mapping theorem.** In Theorem 59 above we got the formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P.$$

Now that we know what complex logarithm is, we can say that $f'/f = (\operatorname{Ln} f)'$ on each specific branch of Ln. So we cut up the curve γ as in Subsection 13.4.4 and on each piece γ_k with endpoints α_k, β_k we have

$$\int_{\gamma_k} \frac{f'(z)}{f(z)}dz = \operatorname{Ln} f(\beta_k) - \operatorname{Ln} f(\alpha_k) = \Delta_{\gamma_k} \operatorname{Ln} f,$$

where $\Delta_l g$ stands for a change of value of a function g along a curve l, as defined in Subsection 13.4.4. Now sum up the corresponding formulas for all k. We get

$$\int_\gamma \frac{f'(z)}{f(z)}dz = \Delta_\gamma \operatorname{Ln} f.$$

Lucky for us, Ln is a relatively simple function. Inspecting the equality $e^{x+iy} = e^x(\cos y + i \sin y)$, we get that

$$\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z.$$

Function $\ln|z|$ is single-valued, so $\Delta_\gamma \ln|f| = 0$, therefore

$$\Delta_\gamma \operatorname{Ln} f = i\Delta_\gamma(\operatorname{Arg} f).$$

To sum up, $\int_\gamma \frac{f'(z)}{f(z)}dz$ is i times the angle that f travels while z goes along γ.

Notice that if γ is a closed curve then $\Delta_\gamma \operatorname{Ln} f$ is a multiple of $2\pi i$ (since values of Ln at the same point can only differ by $2\pi i n$). The integer number $\frac{1}{2\pi i}\Delta_\gamma \operatorname{Ln} f =$

$\frac{1}{2\pi}\Delta_\gamma(\operatorname{Arg} f)$ is called the *index* of γ w.r.t. 0 (or winding number of γ w.r.t. 0). Notation: $\operatorname{ind}_0(\gamma)$. In plain terms, this is the number of loops (full turns) that γ makes around 0.

Theorem 60. (Argument Principle) *Let G be a domain, γ a simple rectifiable closed curve contained in G together with its interior, $\gamma \cup I(\gamma) \subseteq G$. Let f a function meromorphic on G, without zeros or poles on γ. Let N be the number of zeros of f in $I(\gamma)$ and P be the number of poles of f in $I(\gamma)$ (counting multiplicity). Then*

$$N - P = \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} f = \operatorname{ind}_0(f(\gamma)).$$

Proof. By Theorem 59, we know that

$$N - P = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\Delta_\gamma(\operatorname{Arg} f).$$

The latter is, by definition, equal to $\operatorname{ind}_0(f(\gamma))$. □

By N_f we denote the number of zeros of a function f (counting multiplicity).

Theorem 61. (Rouché Theorem) *Let G be a domain, γ a simple rectifiable closed curve contained in G together with its interior, $\gamma \cup I(\gamma) \subseteq G$. Let functions F, g be analytic in G, and let $|F(z)| > |g(z)|$ for all $z \in \gamma$. Then $F + g$ has the same number of zeros in $I(\gamma)$ as F does:*

$$N_{F+g} = N_F$$

Proof. Since $|F(z)| > |g(z)|$ on γ, it is also > 0, so F does not have any zeros on γ. Since $|g| < |F|$ on γ, neither does F, so we can apply Argument Principle:

$$\begin{aligned} N_{F+g} &= \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg}(F+g) = \\ &= \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg}\left(F\left(1+\frac{g}{F}\right)\right) = \\ &= \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} F + \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg}\left(1+\frac{g}{F}\right). \end{aligned}$$

Note that $\Delta_\gamma \operatorname{Arg}\left(1+\frac{g}{F}\right) = 0$ since $\left|\frac{g}{F}\right| < 1$. Therefore

$$N_{F+g} = \frac{1}{2\pi}\Delta_\gamma \operatorname{Arg} F = N_F.$$

□

Remark. Note that we haven't really used analyticity of f other than to apply Argument Principle. Therefore, the same statement is true for meromorphic functions and the value $N - P$, that is $N_{F+g} - P_{F+g} = N_F - P_F$.

Remark. Another observation is that asking $|F| > |g|$ is a bit more than it is necessary for the proof to work. Indeed, we only need $1 + \frac{g}{F}$ to not go around the origin, while we condition of the theorem provides a stronger restriction: that $1 + \frac{g}{F}$ stays in the disc $|z - 1| < 1$. The same proof will work if we ask that the value $1 + \frac{g}{F} = \frac{F+g}{F}$ is simply never negative. Note that a ratio of two complex numbers is a negative real if and only if their arguments differ by π. In our case it just means that we want to forbid the equality $|F+g| + |F| = |g|$ (see Fig. 7. Since by triangle inequality $|F+g| + |F| \geq |g|$, the requirement becomes $|F+g| + |F| > |g|$ at all

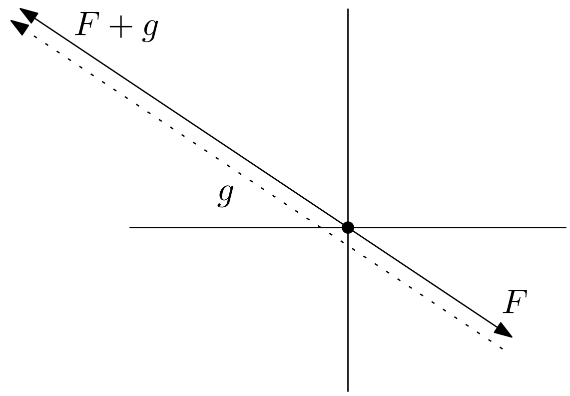

FIGURE 7. Forbidden sum in Rouché theorem.

points of γ. This is easier to satisfy than $|F| > |g|$ because of the extra term $|F+g|$ in the left hand side.

EXAMPLE 1. *Find number of roots of the equation*

$$z^{2017} + 7z^8 - e^z + 3i = 0$$

of absolute value less than 1.

Put $F(z) = 7z^8$, $g(z) = z^{2017} - e^z + 3i$, $\gamma : |z| = 1$. Then $|F(z)| = 7$ on γ, and $|g(z)| \leq 1 + e + 3 < 7$ on γ. By Rouché Theorem, the function $z^{2017} + 7z^8 - e^z + 3i$ has the same number of zeros inside γ as $F(z) = 7z^8$, i.e. 8 zeros.

EXAMPLE 2. *Find number of roots of the equation*

$$z^{2017} + 7z^8 - e^z + 3i = 0$$

of absolute value less than 2.

Put $F(z) = z^{2017}$, $g(z) = 7z^8 - e^z + 3i$, $\gamma : |z| = 2$. Then $|F(z)| = 2^{2017} > 10000$ on γ, and $|g(z)| \leq 7 \cdot 2^8 + e^2 + 3 < 10000$ on γ. By Rouché Theorem, the function $z^{2017} + 7z^8 - e^z + 3i$ has the same number of zeros inside γ as $F(z) = z^{2017}$, i.e. 2017 zeros.

Observe also that by the preceding example, we also know that precisely $2017 - 8 = 2009$ of those zeros are in the annulus $1 < |z| < 2$.

EXAMPLE 3. *Find number of roots of the same equation*

$$z^6 - 4z^3 + 7z^2 - 3 = 0$$

of absolute value less than 2.

Put $F(z) = z^6$, $g(z) = -4z^3 + 7z^2 - 1$, $\gamma : |z| = 2$. Then $|F(z)| = 2^6 = 64$ on γ, and $|g(z)| \leq 32 + 28 + 3 < 64$ on γ. By Rouché Theorem, the function $z^6 - 4z^3 + 7z^2 - 3$ has the same number of zeros inside γ as $F(z) = z^6$, i.e. 6 zeros.

Note that Example 3 can be done without the Rouché theorem, because by a similar estimate, the term z^{2017} dominates other terms on the annulus $|z| \geq 2$, not just on the circle $|z| = 2$. Since the function is a degree 6 polynomial, we know that there are 6 zeros in $\mathbb{C}$. Since there are none in the annulus $|z| \geq 2$, it follows that all 6 are in the disk $|z| < 2$..

Rouché theorem also gives an easy way to prove the Fundamental Theorem of Algebra. Indeed, for a polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \ n \geq 1,$$

set

$$F(z) = z^n \text{ and } g(z) = a_{n-1}z^{n-1} + \ldots + a_1 z + a_0.$$

Since $g(z)$ is of degree $n-1 < n$, for R large enough we have that $|F(z)| > |g(z)|$ provided $|z| = R$. Therefore, $f(z)$ has as many zeros in the disc $|z| < R$ as $F(z) = z^n$ does, that is n zeros.

Yet another corollary of Roucbé theorem is the following statement.

Theorem 62. (Open mapping theorem, open mapping principle) *Analytic non-constant function defines an open mapping, i.e. if G is a domain and $f \neq \text{const}$ is analytic on G, then $f(G)$ is also a domain.*

Proof. $f(G)$ is connected since G is connected and f is continuous. So we only need to show that $f(G)$ is open. Suppose $b \in f(G)$, that is there is an $a \in G$ s.t. $f(a) = b$. Since $f \neq 0$, then by Interior Uniqueness Theorem 44, there is a closed neighborhood $\overline{B}_\varepsilon(a)$ such that $f(z) \neq b$ for any $z \in \overline{B}_\varepsilon(a), z \neq a$. (Otherwise we would be able to construct a sequence $z_n \to a$ such that $f(z_n) = b$ and that would mean that $f \equiv b$). Put

$$\delta = \min_{|z-a|=\varepsilon} |f(z) - b| > 0.$$

Consider an open circle of radius δ centered at b. Suppose a point w lies in this circle. Show that w has a preimage in G. Indeed, write

$$f(z) - w = (f(z) - b) + (b - w).$$

Put $F(z) = f(z) - b$, $g(z) = b - w$ in Rouché Theorem. Then $|g| < \delta$ (because w is inside a circle of radius δ), while $|F| \geq \delta$ for $|z-a| = \varepsilon$. Then $N_{F+g} = N_F$. We know that $f(z) - b$ has at least one zero, so N_{F+g} is also ≥ 1, which precisely means that w has a preimage. We showed that $f(G)$, together with a point $b \in f(G)$, contains its open neighborhood of radius $\delta > 0$. Therefore, $f(G)$ is open. $\square$

Open mapping theorem is quite a strong statement. For example, Maximum Modulus Principle follows immediately from open mapping theorem.

14.2. **Bonus track: Riemann zeta-function and the statement of Riemann Hypothesis.** *This section is not included in Final.*

In the Homework Assignment 13, we computed the sums

$$\sum_{k=0}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{k^4}.$$

One can define the same sum for an arbitrary exponent $s \in \mathbb{C}$:

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{k^s},$$

where we can define $k^s = e^{s \ln k}$. Note that this series does not always converge. In fact, we can only guarantee that it converges if $\operatorname{Re} s > 1$. Indeed, if $\operatorname{Re} s \geq \sigma_0 > 1$, then the series is dominated by

$$\sum_{k=0}^{\infty} \frac{1}{k^{\sigma_0}},$$

so by Weierstrass M-test (Theorem 34) the series uniformly converges in the region $\operatorname{Re} s \geq \sigma_0$ for any $\sigma_0 > 1$. Further, by Weierstrass theorem 39, the sum is analytic on the domain $\operatorname{Re} s > 1$.

The function $\zeta(s)$ defined above is called *Riemann zeta-function*. We can now "state" the famous Riemann Hypothesis.

Riemann Hypothesis. *All zeros of Riemann zeta-function $\zeta(s)$ in the right half-plane $\operatorname{Re} s > 0$ are located on the vertical line $\operatorname{Re} s = \frac{1}{2}$.*
(See also restatement in the end of this section.)

Of course, for this to make sense we need to define $\zeta(s)$ on a region that at least contains that line. Below we extend ζ to the domain $\operatorname{Re} s > 0$.

One can view the following lemma as a discrete analogue of integration by parts.

Lemma 8. *(Abel transformation) Let (a_n) be a sequence in $\mathbb{C}$, $x \in \mathbb{R}$, $x > 1$. Put*

$$A(x) = \sum_{n \leq x} a_n.$$

Let $g(t) : [1, +\infty) \to \mathbb{C}$ be differentiable. Then

$$\sum_{n \leq x} a_n g(n) = A(x)g(x) - \int_1^x A(t)g'(t)dt.$$

Proof. Denote $B(x) = \sum_{n \leq x} a_n g(n)$.

Let $x = N \in \mathbb{Z}$ first. We have

$$\begin{aligned}
B(N) &= \sum_{n=1}^{N} a_n g(n) = \sum_{n=1}^{N} (A(n) - A(n-1))g(n) = \\
&= \sum_{n=1}^{N} A(n)g(n) - \sum_{n=1}^{N} A(n-1)g(n) = \\
&= \sum_{n=1}^{N} A(n)g(n) - \sum_{n=1}^{N-1} A(n)g(n+1) = \\
&= A(N)g(N) - \sum_{n=1}^{N-1} A(n)(g(n+1) - g(n)),
\end{aligned}$$

since $A(0) = 0$. Then we get

$$\begin{aligned}
B(N) &= A(N)g(N) - \sum_{n=1}^{N-1} A(n)(g(n+1) - g(n)) = \\
&= A(N)g(N) - \sum_{n=1}^{N-1} A(n) \int_n^{n+1} g'(t)dt = \\
&= A(N)g(N) - \int_1^N A(t)g'(t)dt,
\end{aligned}$$

so the case $x = N$ is done. Now for arbitrary $x > 1$, put $x = N + \{x\}$, where $\{x\} = x - [x]$ is the fractional part of x, and plug it in the required equality (exercise). $\square$

Note that if we additionally know that $A(x)g(x) \to 0$ as $x \to \infty$, then by Lemma 8 we get

$$\sum_{n=1}^{\infty} a_n g(n) = -\int_1^{\infty} A(t)g'(t)dt, \tag{17}$$

where the latter is an improper Riemann integral.

Now apply this to the sum that defines zeta-function: put $a_n = 1$ and $g(x) = (x+1)^{-s}$. Then $A(x) = [x]$, so $A(x) \leq x$ and

$$\lim_{x\to\infty} A(x)g(x) = \lim_{x\to\infty} \frac{x}{(x+1)^s} = 0$$

if $\operatorname{Re} s > 1$. So we have

$$\zeta(s) = \frac{1}{1^s} + \sum_{n=1}^{\infty} \frac{1}{(1+n)^s} = 1 + \sum_{n=1}^{\infty} a_n g(n),$$

so applying (17) we get

$$\begin{aligned}
\zeta(s) &= 1 - \int_1^{\infty} A(t)g'(t) = 1 - \int_1^{\infty} [t](-s)(t+1)^{-s-1}dt = \\
&= 1 - \int_0^{\infty} [t](-s)(t+1)^{-s-1}dt = 1 + s\int_1^{\infty} \frac{[t-1]}{t^{s+1}}dt = \\
&= 1 + s\int_1^{\infty} \frac{t-1-\{t\}}{t^{s+1}}dt = \\
&= 1 + s\int_1^{\infty} \frac{dt}{t^s} - s\int_1^{\infty} \frac{1}{t^{s+1}}dt - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}dt = \\
&= 1 + s\tfrac{1}{s-1} - s\tfrac{1}{s} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}dt.
\end{aligned}$$

So we get the expression

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}dt. \tag{18}$$

Note that $\{x\}$ is bounded by 1, so the latter integral converges whenever $\operatorname{Re} s > 0$. Moreover, the latter integral is

$$\int_1^{\infty} \frac{\{t\}}{t^{s+1}}dt = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{t-n}{t^{s+1}}dt.$$

By Weierstrass M-test this series converges uniformly on the region $\operatorname{Re} s \geq \sigma_0 > 0$, since

$$\left| \int_n^{n+1} \frac{t-n}{t^{s+1}}dt \right| \leq \frac{1}{n^{\operatorname{Re} s+1}}.$$

Note further, than each $\int_n^{n+1} \frac{t-n}{t^{s+1}}dt$ is analytic on the domain $\operatorname{Re} s > 0$ (the integral can be evaluated explicitly), so by Weierstrass theorem, the sum of series is analytic function on that domain. Now we see that (18) defines a meromorphic function on the right half plane $\operatorname{Re} s > 0$, and this function coincides with $\sum \frac{1}{n^s}$ on the domain $\operatorname{Re} s > 1$. (Moreover, note that the function ζ has only one pole, at $s = 1$, it's a simple pole with $\operatorname{res}_1 \zeta = 1$.)

So the formula (18) extends ζ to the domain $\operatorname{Re} s > 0$, and now the statement of Riemann Hypothesis makes sense. Note that through a more complicated reasoning one can extend ζ to the whole complex plane $\mathbb{C}$. In that case ζ is still meromorphic with a unique pole at $s = 1$, and essential singularity at ∞. Moreover, ζ has zeros at even negative numbers: $0 = \zeta(-2) = \zeta(-4) = \dots$. These are called *trivial* zeros of ζ. The Riemann Hypothesis states that aside from trivial zeros, all zeros have real part 1/2. (It is not a significant change over the previous statement in the beginning of the Section that we fully understand; but it is usually the way you will see the Riemann Hypothesis stated, so it is worth it to include it here even though we haven't done the work to extend ζ to $\operatorname{Re} s < 0$.)

Riemann Hypothesis. *All nontrivial zeros of Riemann zeta-function $\zeta(s)$ are located on the vertical line* $\operatorname{Re} s = \frac{1}{2}$.

Riemann Hypothesis is one of the most important unsolved problems in mathematics, (arguably) of the same caliber as (solved) Fermat Theorem, (solved) Poincaré Conjecture, **P** versus **NP**, Navier–Stokes existense/smoothness, and few others (see Millennium Prize Problems for the list). Proving or disproving Riemann Hypothesis would give a deep insight into several areas of mathematics, most immediately Number Theory and distribution of primes.